On subhomogeneous indefinite p-Laplace equations in the supercritical spectral interval

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Abstract

We study the existence, multiplicity, and certain qualitative properties of solutions to the zero Dirichlet problem for the equation $-\Delta_p u = \lambda |u|^{p-2}u + a(x)|u|^{q-2}u$ in a bounded domain $\Omega \subset \mathbb{R}^N$, where $1 < q < p$, $\lambda \in \mathbb{R}$, and a is a sign-changing weight function. Our primary interest concerns ground states and nonnegative solutions which are positive in $\{x \in \Omega : a(x) > 0\}$, when the parameter λ lies in a neighborhood of the critical value $\lambda^* := \inf \left\{ \int_\Omega |\nabla u|^p \, dx / \int_\Omega |u|^p \, dx : u \in W_0^{1,p}(\Omega) \setminus \{0\},\ \int_\Omega a|u|^q \, dx \geq 0 \right\}$. Among main results, we show that if $p > 2q$ and either $\int_\Omega a\varphi_p^q \, dx = 0$ or $\int_\Omega a\varphi_p^q \, dx > 0$ is sufficiently small, then such solutions do exist in a *right* neighborhood of λ^*. Here φ_p is the first eigenfunction of the Dirichlet p-Laplacian in Ω. This existence phenomenon is of a purely subhomogeneous and nonlinear nature, since either in the superhomogeneous case $q > p$ or in the sublinear case $q < p = 2$ the nonexistence takes place for any $\lambda \geq \lambda^*$. Moreover, we prove that if $p > 2q$ and $\int_\Omega a\varphi_p^q \, dx > 0$ is sufficiently small, then there exist *three* nonzero nonnegative solutions in a *left* neighborhood of λ^*, two of which are strictly positive in $\{x \in \Omega : a(x) > 0\}$.

Keywords: p-Laplacian, subhomogeneous, sublinear, existence, nonexistence, three solutions, ground states, least energy solutions, positive solutions, fibered functional, improved Picone inequality.

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1. Introduction

In the present work, we study the boundary value problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + a(x)|u|^{q-2}u & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where the p-Laplace operator Δ_p with $p > 1$ acts formally as $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $\lambda \in \mathbb{R}$ is a parameter, and $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 1$. Unless explicitly stated otherwise,

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we always assume throughout the paper that $1 < q < p$ and the weight function $a \in L^\infty(\Omega)$ satisfies

$$\Omega_a^+ \neq \emptyset, \quad \Omega_a^- \neq \emptyset, \quad \text{and} \quad a \leq 0 \text{ a.e. in } \Omega \setminus \Omega_a^+, \tag{1.1}$$

where

$$\Omega_a^+ := \cup\{U \subset \Omega : \ U \text{ is open and } a > 0 \text{ a.e. in } U\},$$
$$\Omega_a^- := \cup\{U \subset \Omega : \ U \text{ is open and } a < 0 \text{ a.e. in } U\}.$$

Because of these assumptions, the problem (P_λ) is called *subhomogeneous* ($q < p$) and *indefinite* ($\Omega_a^\pm \neq \emptyset$). Nevertheless, a few of our results remain valid when a is a.e. sign-constant.

We remark that the assumptions on a can be weakened, although we do not pursue this aim here in order to avoid technical complications which might obscure more essential aspects of the work. In the model case of the *continuous* weight a, the sets $\Omega_a^\pm$ can be equivalently defined as $\Omega_a^\pm = \mathrm{Int}(\overline{\{x \in \Omega : \ \pm a(x) > 0\}})$, and the third assumption in (1.1) is automatically satisfied.

The problem (P_λ) is understood in the weak sense. Namely, we say that $u \in W_0^{1,p}(\Omega)$ is a (weak) solution of (P_λ) if the equality

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx = \lambda \int_\Omega |u|^{p-2} u \varphi \, dx + \int_\Omega a|u|^{q-2} u \varphi \, dx$$

holds for all $\varphi \in W_0^{1,p}(\Omega)$. It is not hard to see that solutions are precisely critical points of the energy functional $I_\lambda \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ defined as

$$I_\lambda(u) := \frac{1}{p} E_\lambda(u) - \frac{1}{q} \int_\Omega a|u|^q \, dx, \quad \text{where} \quad E_\lambda(u) := \int_\Omega |\nabla u|^p \, dx - \lambda \int_\Omega |u|^p \, dx. \tag{1.2}$$

Remark 1.1. Any solution of (P_λ) belongs to $C_0^{1,\beta}(\Omega)$ with some $\beta \in (0,1)$. In fact, if u is a solution of (P_λ), then $u \in L^\infty(\Omega)$, which can be shown by the standard Moser iteration process (see, e.g., [35, Appendix A]). This implies $u \in C_0^{1,\beta}(\Omega)$ by the regularity results of [19] or [44]. If, in addition, Ω is of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$, then $u \in C_0^{1,\beta}(\overline{\Omega})$, see [33].

The distinguishing feature of (P_λ) to comprise a mixture of the subhomogeneous (even non-Lipschitz when $q < 2$) and indefinite natures made this simply looking problem a subject of considerable interest over the last thirty years, and several important contributions to its understanding have been provided only recently, even in the case $p = 2$ and $\lambda = 0$. In particular, thanks to the assumption $q < p$ and the presence of the sign-changing weight a, *nonnegative solutions of* (P_λ) *do not obey, in general, the strong maximum principle.* (See, for instance, [40] for a comprehensive summary of maximum principles.) As a result, an emergence of the so-called *dead core* solutions, i.e., nonzero solutions vanishing in a subdomain of Ω, can be observed. We refer the reader to, e.g., [6, 16, 32] for a deeper discussion which includes explicit constructions of such solutions, their qualitative properties, and description of earlier results in this direction.

Apart from the dead core solutions, two other classes of solutions of (P_λ) are of significant importance:

1. Ground states and nonnegative ground states.

Definition 1.2. We say that a nonzero critical point u of I_λ is a *ground state* (or, equivalently, a *least energy solution*) of (P_λ) if $I_\lambda(u) \le I_\lambda(v)$ for any nonzero critical point v of I_λ.

2. Solutions which are (strictly) positive[1] in Ω_a^+.

In general, ground states may not always be either positive or negative in Ω_a^+. Indeed, (P_λ) might even possess *sign-changing* ground states, as it follows, e.g., from [32, Theorem 1.8] (see, more precisely, [32, Remark 1.9] for the construction of a two-bumps solution of (P_λ) in 1D case, one bump of which might be reflected over the x-axis to make this solution sign-changing). On the other hand, under certain assumptions, there exist nonnegative mountain pass solutions of (P_λ) which are positive in Ω_a^+, but they are not ground states, see, e.g., [32, Theorem 1.4]. Thus, the classes of ground states and solutions which are positive in Ω_a^+ are independent. These two classes of solutions are the main objects of our study. Several known results on the existence of such solutions with respect to the parameter λ are collected in Section 1.1.

The combination of subhomogeneous and indefinite natures suggests the consideration of nonnegative solutions of the problem (P_λ) in the following three ranges of the parameter:

1. Nonpositive values of λ, with the special emphasis on the case $\lambda = 0$. Such values of λ allow, in particular, a deeper investigation of the formation of dead cores or positivity, and the study of uniqueness issues. We refer to the series of articles [29, 30, 31, 32] and references therein.

2. "Middle" values of λ. In this range, the primary interest is to consider the existence of negative energy ground states and the multiplicity of solutions which are positive in Ω_a^+. The results of the present work correspond to such values of λ, see Sections 1.1 and 1.2 for an overview.

3. "Large" values of λ. In this range, there are no solutions positive in Ω_a^+ (see Proposition 5.2 below). However, nonzero solutions which vanish in Ω_a^+ might exist. The existence and properties of such solutions, positive energy ground states, and so-called compact support solutions are of the main importance. We refer to [5, 17, 18, 24] for the consideration of these and related issues.

Let us mention that the problem (P_λ) in the superhomogeneous regime $q > p$ is somewhat more developed.[2] The strong maximum principle holds in this case, which yields the positivity of any nonzero nonnegative solution. In particular, there are no nonnegative dead core solutions, and there are no positive solutions for sufficiently large λ. Although the base-level multiplicity information for $q > p$ and $q < p$ for the "middle" values of λ looks similar, the structure and properties of the corresponding solution sets are completely different. The difference is amplified by our main results stated in Section 1.2. We refer the reader to the list of classical works [2, 3, 7, 22, 26, 38] and to a more recent article [27] on the problem (P_λ) in

[1] Throughout this work, the words "positive" and "negative" mean "> 0" and "< 0", respectively. The word "strictly" will be used occasionally for accentuation and clarification.

[2] When commenting on the superhomogeneous case $q > p$, we always assume that $q < p^*$, where p^* is the critical Sobolev exponent for $N \ge 3$, and $p^* = +\infty$ for $N = 1, 2$.

the superhomogeneous case $q > p$. In the following subsections, we will comment on several known results from these works in more details.

Finally, we mention that even when the weight a is sign-constant, various recent contributions on the subhomogeneous problem (P_λ) have been made. In the case of continuous $a \geq 0$, in which the strong maximum principle holds, we refer to [11] for the existence of least energy nodal solutions (a is constant), to [12] for an overview on the corresponding eigenvalue problem ($\lambda = 0$ and $a(x) = \mu$), and to [28] for the existence of infinitely many solutions with negative energy converging to zero (the case of sign-changing a is also covered). On the other hand, in the case $a \leq 0$, the problem (P_λ) has been investigated in, e.g., [5, 17, 18, 24], in the context of study of compact support solutions. Note that such solutions also satisfy (P_λ) with a sign-changing weight a whenever Ω_a^+ does not intersect with their supports.

In the following subsection, we briefly overview several known results, as well as new ones, on the problem (P_λ) which are the basis of our further analysis provided in Section 1.2.

1.1. Behavior in the subcritical spectral interval

Hereinafter, for brevity, we denote by $W_0^{1,p} := W_0^{1,p}(\Omega)$ the standard Sobolev space and by $\|\cdot\|_r$ the standard norm in the Lebesgue space $L^r(\Omega)$, $r \in [1, +\infty]$. The first eigenvalue of the Dirichlet p-Laplacian is denoted by

$$\lambda_1(p) := \inf \left\{ \frac{\|\nabla u\|_p^p}{\|u\|_p^p} : \ u \in W_0^{1,p} \setminus \{0\} \right\}.$$

This eigenvalue is simple and the corresponding first eigenfunction φ_p has a strict constant sign in Ω, see, e.g., [4, 40]. Moreover, φ_p belongs to $C_0^{1,\beta}(\Omega)$, $\beta \in (0,1)$, see Remark 1.1. Hereinafter, we assume, without loss of generality, that $\varphi_p > 0$ in Ω and $\|\nabla \varphi_p\|_p = 1$.

Noting that the energy functional I_λ defined by (1.2) is bounded from below and coercive when $\lambda < \lambda_1(p)$, it is not hard to show that in this case I_λ has a global minimizer, this minimizer has negative energy, it is a ground state of (P_λ), and any nonnegative minimizer is positive in Ω_a^+, cf. [32, Section 1.2]. On the other hand, by taking $u = t\varphi_p$ and letting $t \to +\infty$, we see that $\inf\{I_\lambda(u) : u \in W_0^{1,p}\} = -\infty$ when $\lambda > \lambda_1(p)$, and hence no global minimizer exists. Consequently, ground states of (P_λ) cannot be characterized as global minimizers of I_λ for $\lambda > \lambda_1(p)$. In the borderline case $\lambda = \lambda_1(p)$, the existence of global minimizers is a more subtle issue which depends on the settings of the problem. We provide a corresponding result in Section 1.2.

In order to find a possibly larger spectral interval of the existence of solutions to (P_λ), it is natural to investigate minimizers of I_λ over a subset of $W_0^{1,p}$ described by the *Nehari manifold*

$$\mathcal{N}_\lambda := \left\{ u \in W_0^{1,p} \setminus \{0\} : \ \langle I_\lambda'(u), u \rangle = 0 \right\} = \left\{ u \in W_0^{1,p} \setminus \{0\} : \ E_\lambda(u) = \int_\Omega a|u|^q \, dx \right\}. \tag{1.3}$$

Consider the corresponding minimal level of I_λ:

$$M(\lambda) := \inf\{I_\lambda(u) : \ u \in \mathcal{N}_\lambda\}. \tag{1.4}$$

The following result asserts that ground states of (P_λ) can be characterized as minimizers of $M(\lambda)$ whenever $M(\lambda)$ is attained, cf. [32, Theorem 1.4 (1)].

Proposition 1.3. *Let $\lambda \in \mathbb{R}$ be such that $M(\lambda)$ is attained. Then any corresponding minimizer u is a ground state of (P_λ), $I_\lambda(u) < 0$, u is a local minimum point of I_λ, and either $u > 0$ or $u < 0$ in every connected component of Ω_a^+. Moreover, u is a global minimum point of I_λ provided $\lambda \le \lambda_1(p)$.*

Remark 1.4. Evidently, if u is a minimizer of $M(\lambda)$, then so is $|u|$. That is, the attainability of $M(\lambda)$ implies the existence of a nonnegative minimizer which is positive in Ω_a^+. On the other hand, $M(\lambda)$ might possess sign-changing minimizers, as we indicated above.

Proposition 1.3 motivates the investigation of assumptions under which $M(\lambda)$ is attained. For this purpose, we introduce the following critical value of the parameter λ which plays a significant role throughout the work:

$$\lambda^* := \inf\left\{\frac{\|\nabla u\|_p^p}{\|u\|_p^p} :\ u \in W_0^{1,p} \setminus \{0\},\ \int_\Omega a|u|^q\,dx \ge 0\right\}. \tag{1.5}$$

We readily have $\lambda_1(p) \le \lambda^*$, and the simplicity of $\lambda_1(p)$ yields $\lambda_1(p) < \lambda^*$ if and only if $\int_\Omega a\varphi_p^q\,dx < 0$, see Proposition 2.1 below for details. The critical value λ^* has the property that the set $\mathcal{N}_\lambda$ is a C^1-manifold of codimension 1 for any $\lambda < \lambda^*$, see, e.g., [13, 41]. In particular, the following information can be obtained (see, e.g., [32, Lemma 2.3 (1) and Remark 2.4], whose arguments remain valid under our assumptions on a.)

Theorem 1.5. *The following assertions hold:*

(i) *If $\lambda < \lambda^*$, then $M(\lambda) \in (-\infty, 0)$ and it is attained.*

(ii) *If $\lambda > \lambda^*$, then $M(\lambda) = -\infty$.*

Notice that $M(\lambda)$ can be also characterized as

$$M(\lambda) = M^+(\lambda) := \inf\{I_\lambda(u) :\ u \in \mathcal{N}_\lambda \cap \mathcal{A}^+\},$$

where

$$\mathcal{A}^+ := \left\{u \in W_0^{1,p} :\ \int_\Omega a|u|^q\,dx > 0\right\}, \tag{1.6}$$

see Remark 2.5 below. In the following proposition, we collect a few qualitative results on the behavior of $M(\lambda)$. We refer the reader to Section 4 for additional information on the behavior of $M(\lambda)$ and the corresponding minimizers.

Proposition 1.6. *The extended function $M : \mathbb{R} \mapsto \mathbb{R} \cup \{-\infty\}$ has the following properties:*

(i) *M is nonincreasing on $\mathbb{R}$ and decreasing on $(-\infty, \lambda^*]$.*

(ii) *M is continuous on $(-\infty, \lambda^*)$.*

(iii) *$M(\lambda) \to M(\lambda^*)$ as $\lambda \to \lambda^* - 0$.*

In view of the discussion on the global minimizers above, Theorem 1.5 (i) provides nontrivial information if $\lambda_1(p) < \lambda^*$, i.e., when $\int_\Omega a\varphi_p^q\,dx < 0$. Moreover, if $\int_\Omega a\varphi_p^q\,dx < 0$, then for any $\lambda \in (\lambda_1(p), \lambda^*)$ there exists another nonnegative solution of (P_λ) which is positive in Ω_a^+. This solution has the least energy among all positive energy solutions and can be characterized as a minimizer of

$$M^-(\lambda) := \inf\{I_\lambda(u) : \; u \in \mathcal{N}_\lambda \cap \mathcal{A}^-\}, \tag{1.7}$$

where

$$\mathcal{A}^- := \left\{ u \in W_0^{1,p} : \; \int_\Omega a|u|^q \, dx < 0 \right\}. \tag{1.8}$$

More precisely, the following result is given by [32, Theorem 1.4 (2)], whose proof remains valid under our assumptions on a.

Theorem 1.7. *Let $\int_\Omega a\varphi_p^q \, dx < 0$ and $\lambda \in (\lambda_1(p), \lambda^*)$. Then $M^-(\lambda) > 0$, it is attained, and there exists a nonnegative minimizer of $M^-(\lambda)$ which is positive in Ω_a^+. Moreover, $M^-(\lambda)$ coincides with a mountain pass value of I_λ such that*

$$M^-(\lambda) = \inf_{u \in W_0^{1,p} \setminus \{0\}} \sup_{t>0} I_\lambda(tu).$$

Remark 1.8. Under the assumptions of Theorem 1.7, any minimizer of $M^-(\lambda)$ is a saddle point of I_λ (i.e., neither a local minimum nor local maximum). The proof of this fact can be obtained along the same lines as the proof of [8, Proposition 2.9 (ii)].

Let us collect a few qualitative results on the behavior of $M^-(\lambda)$, see also Section 4 for further properties of $M^-(\lambda)$.

Proposition 1.9. *Let $\int_\Omega a\varphi_p^q \, dx < 0$. Then the following assertions hold:*

(i) *M^- is nonincreasing on $(\lambda_1(p), +\infty)$ and decreasing on $(\lambda_1(p), \lambda^*]$.*

(ii) *M^- is continuous on $(\lambda_1(p), \lambda^*)$.*

(iii) *$M^-(\lambda) \to +\infty$ as $\lambda \to \lambda_1(p) + 0$.*

(iv) *$M^-(\lambda) \to M^-(\lambda^*)$ as $\lambda \to \lambda^* - 0$.*

(v) *$M^-(\lambda) = 0$ for any $\lambda > \lambda^*$.*

Remark 1.10. In the case $\int_\Omega a\varphi_p^q \, dx \geq 0$, it can be shown that for any $\lambda \in \mathbb{R}$ one has either $\mathcal{N}_\lambda \cap \mathcal{A}^- = \emptyset$ (and hence $M^-(\lambda)$ is undefined) or $M^-(\lambda) = 0$, see Lemma 2.4 and Remark 4.2.

Propositions 1.6 and 1.9, together with other results from Section 4, supplement the information on $M^\pm$ from [13, Section 4] and [32, Section 4], and we refer the reader to these works for additional results on the attainability and qualitative properties of $M^\pm$ and their minimizers. Let us mention that there are several other definitions of the minimization problems $M^\pm$ equivalent to (1.4) and (1.7). For instance, the authors of [32] use the minimization over $\mathcal{A}^\pm$ (without explicit reference to $\mathcal{N}_\lambda$), while the authors of [41] deal with definitions like (1.4) and (1.7), but containing the truncated integrals $\int_\Omega u_+^p \, dx$ and $\int_\Omega a u_+^q \, dx$ instead of their untruncated versions $\int_\Omega |u|^p \, dx$ and $\int_\Omega a|u|^q \, dx$. Here, $u_+ := \max\{u, 0\}$. It is not hard to show that all such definitions coincide in the sense that they describe the same critical levels. We give a few rigorous results in this direction in Section 2.4.

It is important to remark that Theorem 1.5 does not provide an answer to the attainability of $M(\lambda^*)$. In general, information about the existence of nonnegative solutions to (P_λ) in the supercritical spectral interval $[\lambda^*, +\infty)$ is very limited. We refer the reader to [28, 36] for the existence of abstract solutions to (P_λ) (without information on the sign) for all λ, and to [18] for the existence of nonnegative solutions with compact support in Ω_a^- for sufficiently large λ (see Proposition 2.19 below). Several nontrivial results on the existence of nonnegative solutions in a right neighborhood of λ^* have been obtained recently in [41]. The authors of [41] develop a theory applicable to general variational functionals consisting of two homogeneous parts obeying certain assumptions. In the case of the problem (P_λ), these assumptions are satisfied only when $\int_\Omega a\varphi_p^q \, dx < 0$. The main aim of the present work is to contribute to the available theory of the existence and multiplicity of nonnegative solutions to (P_λ) by studying in more details the supercritical spectral interval $[\lambda^*, +\infty)$ for all signs of $\int_\Omega a\varphi_p^q \, dx$. In particular, we show that if $p > 2q$ and either $\int_\Omega a\varphi_p^q \, dx = 0$ or $\int_\Omega a\varphi_p^q \, dx > 0$ is "sufficiently small", then the problem (P_λ) exhibits nontrivial existence and multiplicity phenomena which are impossible in the cases $q > p$ and $q < p = 2$.

Precise statements of our main results are given in the following subsection.

1.2. Behavior in the supercritical spectral interval. Statements of the main results

Let us recall that the critical value λ^* defined by (1.5) is the threshold dividing the existence and nonexistence of minimizers of $M(\lambda)$, see Theorem 1.5. Our first main result describes the attainability of $M(\lambda^*)$ with respect to the sign of $\int_\Omega a\varphi_p^q \, dx$. The second assertion of this theorem gives the most essential contribution.

Theorem 1.11. *The following assertions hold:*

(i) *Let $\int_\Omega a\varphi_p^q \, dx < 0$. Then $M(\lambda^*) \in (-\infty, 0)$ and it is attained.*

(ii) *Let $\int_\Omega a\varphi_p^q \, dx = 0$. Assume that $\partial\Omega$ is C^2-smooth and connected provided $p \geq 2q$ and $N \geq 2$. Then $M(\lambda^*) \in (-\infty, 0)$ if and only if $p \geq 2q$. Furthermore, if $p > 2q$, then $M(\lambda^*)$ is attained.*

(iii) *Let $\int_\Omega a\varphi_p^q \, dx > 0$. Then $M(\lambda^*) = -\infty$.*

Remark 1.12. The assertion (i) of Theorem 1.11 can be found in [41, Corollary 3.5] under slightly different, but same in essence, assumptions on (P_λ). The assertion (iii) of Theorem 1.11 is in agreement with [13, Theorem 4].

Remark 1.13. The assumptions on $\partial\Omega$ required in Theorem 1.11 (ii) are only needed to use the improved Poincaré inequality from [23] in the proof. As a consequence, if the result of [23, Theorem 1.1] holds under weaker assumptions on $\partial\Omega$, then the requirements of the assertion (ii) can be generalized accordingly.

Remark 1.14. Recall that $\lambda^* = \lambda_1(p)$ whenever $\int_\Omega a\varphi_p^q \, dx \geq 0$, see Proposition 2.1. Thus, a ground state at $\lambda = \lambda_1(p)$ provided by Theorem 1.11 (ii) for $p > 2q$ is the global minimum point of I_λ, see Proposition 1.3. The attainability of $M(\lambda^*)$ in the case $\int_\Omega a\varphi_p^q \, dx = 0$ and $p = 2q$ remains unknown to us.

The main question arising from Theorems 1.5 and 1.11 is whether λ^* is a terminal point for the existence of nonnegative ground states and nonnegative solutions which are positive in Ω_a^+, or these solutions can be obtained in a right neighborhood of λ^*. Let us mention, for comparison, that in the superhomogeneous regime $q > p$ the answer depends solely on the sign of $\int_\Omega a\varphi_p^q\,dx$. Namely, λ^* *is* the terminal point if and only if $\int_\Omega a\varphi_p^q\,dx \geq 0$, see, e.g., [25]. The situation in the subhomogeneous regime $q < p$ appears to be more delicate since not only the sign of $\int_\Omega a\varphi_p^q\,dx$ matters, but also a nontrivial relation between the exponents p and q starts to play a significant role. In Theorems 1.16, 1.19, and Theorem 1.23 below, we provide two groups of opposite results in this direction, for different relations between p and q. These theorems are among our main results.

Since our primary interest concerns nonnegative solutions of (P_λ), it will be convenient to work with the following *truncated* energy functional:

$$\widetilde{I}_\lambda(u) := \frac{1}{p}\,\widetilde{E}_\lambda(u) - \frac{1}{q}\int_\Omega a u_+^q\,dx, \quad \text{where} \quad \widetilde{E}_\lambda(u) := \|\nabla u\|_p^p - \lambda\|u_+\|_p^p. \tag{1.9}$$

Here, we denote $u_\pm = \max\{\pm u, 0\}$, i.e., $u = u_+ - u_-$. Notice that $\widetilde{I}_\lambda \in C^1(W_0^{1,p}, \mathbb{R})$ and $\widetilde{I}_\lambda$ is weakly lower semicontinuous. Any critical point $u \in W_0^{1,p}$ of $\widetilde{I}_\lambda$ is a *nonnegative* solution to (P_λ), i.e., $u_- \equiv 0$ in Ω, which follows from the fact that $0 = \langle \widetilde{I}_\lambda'(u), u_-\rangle = -\|\nabla u_-\|_p^p$. We consider the corresponding truncated Nehari manifold:

$$\widetilde{\mathcal{N}}_\lambda := \left\{ u \in W_0^{1,p} \setminus \{0\} :\ \widetilde{E}_\lambda(u) = \int_\Omega a u_+^q\,dx \right\},$$

and denote by $\widetilde{M}(\lambda)$ the minimal level of $\widetilde{I}_\lambda$ over $\widetilde{\mathcal{N}}_\lambda$, i.e.,[3]

$$\widetilde{M}(\lambda) := \inf\{\widetilde{I}_\lambda(u) :\ u \in \widetilde{\mathcal{N}}_\lambda\}.$$

Hereinafter, we will use the following notion, cf. Definition 1.2.

Definition 1.15. We say that a nonzero critical point u of $\widetilde{I}_\lambda$ is a *least $\widetilde{I}_\lambda$-energy solution* of (P_λ) if $\widetilde{I}_\lambda(u) \leq \widetilde{I}_\lambda(v)$ for any nonzero critical point v of $\widetilde{I}_\lambda$.

In other words, a least $\widetilde{I}_\lambda$-energy solution is a *nonnegative* solution with the least energy *among all nonnegative solutions*, since any nonnegative solution is a critical point of $\widetilde{I}_\lambda$.

The difference between this notion and the notion of ground state of (P_λ) (given by Definition 1.2 via the untruncated functional I_λ) is subtle. Evidently, I_λ has more critical points than $\widetilde{I}_\lambda$ since it also includes sign-changing solutions to (P_λ). In particular, while least $\widetilde{I}_\lambda$-energy solutions are always nonnegative, there might exist sign-changing ground states of (P_λ), see the discussion at the beginning of the paper. Nevertheless, we show in Proposition 2.8 below that these two notions correspond to the same critical level at least when either $M(\lambda)$ or $\widetilde{M}(\lambda)$ is attained. We provide several other results on the relation between I_λ and $\widetilde{I}_\lambda$ in Section 2.4.

The attainability of $M(\lambda^*)$ described in Theorem 1.11 (i), (ii) is a cornerstone for the following two main theorems which contain nontrivial multiplicity information.

Theorem 1.16. *Let one of the following assumptions be satisfied:*

[3] Throughout this work, the diacritic "tilde" over a capital letter always corresponds to the presence of the truncated integrals $\int_\Omega u_+^p\,dx$ and $\int_\Omega a u_+^q\,dx$ instead of their untruncated counterparts $\int_\Omega |u|^p\,dx$ and $\int_\Omega a|u|^q\,dx$.

(I) $\int_\Omega a\varphi_p^q\,dx < 0$.

(II) $\int_\Omega a\varphi_p^q\,dx = 0$, $p > 2q$, *and* $\partial\Omega$ *is* C^2*-smooth and connected provided* $N \geq 2$.

Then there exist Λ *and* Λ^* *such that* $\lambda^* < \Lambda^* \leq \Lambda < +\infty$ *and the following assertions hold:*

(i) *Let* $\lambda \in (\lambda^*, \Lambda)$. *Then* ($P_\lambda$) *possesses a nonnegative solution* u_λ *such that* $u_\lambda > 0$ *in* Ω_a^+ *and* $\widetilde{I}_\lambda(u_\lambda) < 0$. *Moreover,* ($P_\lambda$) *possesses a least* $\widetilde{I}_\lambda$*-energy solution* w_λ *such that* $w_\lambda > 0$ *in some connected component of* Ω_a^+, *and* $\widetilde{I}_\lambda(w_\lambda) \leq \widetilde{I}_\lambda(u_\lambda) < 0$.

(ii) *Let* $\lambda \in (\lambda^*, \Lambda^*)$. *Then* u_λ *is a local minimum point of* $\widetilde{I}_\lambda$, *and* (P_λ) *possesses another nonnegative solution* v_λ *(*$v_\lambda \neq u_\lambda, w_\lambda$*) such that* $v_\lambda > 0$ *in some connected component of* Ω_a^+, v_λ *is a mountain pass critical point of* $\widetilde{I}_\lambda$, *and* $\widetilde{I}_\lambda(w_\lambda) \leq \widetilde{I}_\lambda(u_\lambda) < \widetilde{I}_\lambda(v_\lambda) < 0$.

(iii) *Let* $\lambda > \Lambda$. *Then* (P_λ) *possesses no nonnegative solution which is positive in* Ω_a^+.

Remark 1.17. We conjecture that the solutions u_λ and w_λ obtained in Theorem 1.16 (i) actually satisfy $u_\lambda \equiv w_\lambda$ in Ω, at least for λ close to λ^*. Moreover, we expect that w_λ and v_λ are positive in the whole Ω_a^+. See Figure 1 for a schematic graphical representation of the results of Theorem 1.16.

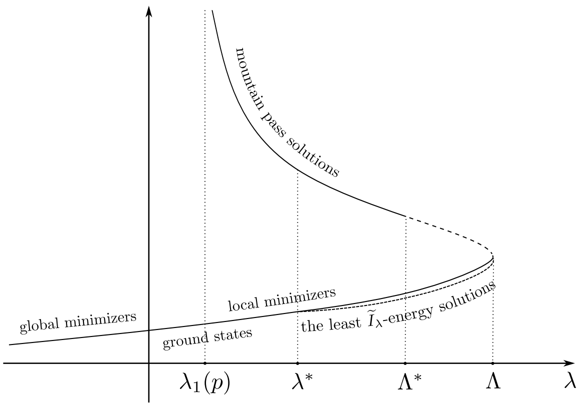

Figure 1: A schematic $L^\infty(\Omega)$-bifurcation diagram provided by Theorems 1.5 and 1.16 under the assumption $\int_\Omega a\varphi_p^q\,dx < 0$.

Remark 1.18. We do not know whether the equality $\Lambda^* = \Lambda$ always takes place, or the inequality $\Lambda^* < \Lambda$ can happen in some regimes. In [1], in the linear case $p = 2$ and under the Neumann boundary conditions, the author obtains the equality under several specific assumptions on the Hölder weight a, namely, that the set $\mathrm{Int}(\{x \in \Omega : a(x) \geq 0\})$ has a finite number of connected components, each of which is C^2-smooth and intersects with the set $\{x \in \Omega : a(x) > 0\}$, see [1, (1.6)-(1.7)]. The proofs of [1] rely in a principal way on the strong comparison principle which is known to be a difficult (and, in many essential cases, open) issue in the general nonlinear case $p > 1$, see, e.g., [15]. The problem is compounded by the fact that nonnegative solutions of (P_λ) do not necessarily obey the Hopf maximum principle, which makes it hard to apply approaches known for "good" nonlinear cases. Thus, the establishment of the equality $\Lambda^* = \Lambda$ under some assumptions for $p > 1$ or under significantly weaker assumptions than [1, (1.5)-(1.7)] for $p = 2$, or the construction of examples when $\Lambda^* < \Lambda$, are interesting open problems.

Take now any nonzero nonnegative $b \in L^\infty(\Omega)$ and define the weight $a_\mu := a + \mu b$ for $\mu > 0$. Consider the boundary value problem analogous to (P_λ):

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + a_\mu(x)|u|^{q-2}u & \text{in } \Omega, \\ \quad\quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P^μ_λ}$$

and denote by $\widetilde{I}^\mu_\lambda$ the corresponding truncated energy functional, i.e., (1.9) with a_μ instead of a. Clearly, if $\int_\Omega a\varphi_p^q\,dx = 0$, then $\int_\Omega a_\mu \varphi_p^q\,dx > 0$. Moreover, we have $\Omega_a^+ \subset \Omega_{a_\mu}^+$.

Theorem 1.19. *Let $p > 2q$, $\partial\Omega$ be C^2-smooth and connected provided $N \geq 2$, the weight a be such that $\int_\Omega a\varphi_p^q\,dx = 0$, and $b \in L^\infty(\Omega) \setminus \{0\}$ be nonnegative a.e. in Ω. Then there exists $\widehat{\mu} > 0$ such that for any $\mu \in [0, \widehat{\mu})$ there exist $\Lambda^* = \Lambda^*(\mu)$ and $\Lambda = \Lambda(\mu)$ satisfying $\lambda_1(p) < \Lambda^* \leq \Lambda < +\infty$ such that the assertions (i)-(iii) of Theorem 1.16 hold for the problem (P^μ_λ) and the corresponding functional $\widetilde{I}^\mu_\lambda$. Moreover, for any $\mu \in (0, \widehat{\mu})$ there exists $\epsilon = \epsilon(\mu) > 0$ such that for any $\lambda \in (\lambda_1(p) - \epsilon, \lambda_1(p))$ the problem (P^μ_λ) possesses at least three distinct nonnegative solutions with negative energy - global minimum, local minimum, and mountain pass critical point of $\widetilde{I}^\mu_\lambda$. The first two critical points are positive in $\Omega^+_{a_\mu}$, while the third one is positive at least in one connected component of $\Omega^+_{a_\mu}$.*

Remark 1.20. Let us notice that $\inf\{\Lambda^*(\mu) : \mu \in [0, \widehat{\mu})\} > \lambda_1(p)$, which follows from the proof of Theorem 1.19. We refer to Figures 2, 3 for a schematic graphical representation of the results of Theorems 1.16 and 1.19.

Remark 1.21. It is interesting to observe that the result of Theorem 1.19 on the existence of two nonnegative solutions which are positive in Ω_a^+ in a left neighborhood of $\lambda_1(p)$ is no longer valid if one replaces the zero Dirichlet boundary conditions in (P_λ) with their Neumann counterpart. Indeed, under the zero Neumann boundary conditions, the *uniqueness* of nonnegative solutions which are positive in Ω_a^+ holds for any $\lambda < 0$, where 0 is the first eigenvalue of the Neumann p-Laplacian, see, e.g., [1, Theorem 1.1] and [32, Theorem 1.1]. On the other hand, there are many results on indefinite subhomogeneous equations as in (P_λ) which are valid for both types of boundary conditions and even for the Robin ones, see [30, 31, 32] for an overview.

Remark 1.22. In essence, the proofs of Theorems 1.16 and 1.19 are based on the observation that the set of nonnegative minimizers of $M(\lambda^*)$ (which is nonempty by Theorem 1.11 (i), (ii)) has a strict local minimum type geometry which is stable under continuous perturbations of $\widetilde{I}_\lambda$. In particular, our arguments on the existence of a local minimum point are not restricted solely to $\widetilde{I}^\mu_\lambda$, but applicable to *any* continuous perturbation of the functional $\widetilde{I}_{\lambda^*}$.

Recently, the literature has been enriched with a few results on the local continuation of the branch of nonnegative solutions to various problems with respect to the parameter beyond a critical value of the type λ^* characterizing the limit of applicability of the Nehari manifold method, see, e.g., [10, 27, 41, 42]. To the best of our knowledge, the first contribution in this direction was made in [27] for the problem (P_λ) in the superhomogeneous regime $q > p$. A

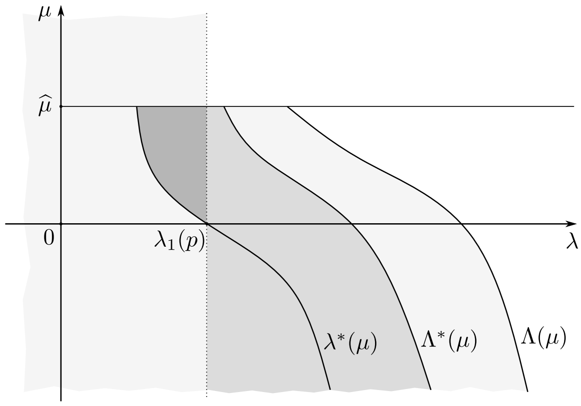

Figure 2: A schematic picture of the regions in the (λ, μ)-plane provided by Theorem 1.16 (for $\mu \leq 0$) and Theorem 1.19 (for $\mu \in (0, \widehat{\mu})$). Light gray - at least one solution. Gray - at least two solutions. Dark gray - at least three solutions.

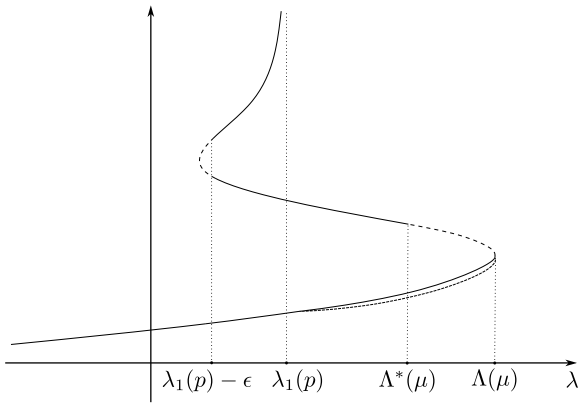

Figure 3: A schematic $L^\infty(\Omega)$-bifurcation diagram provided by Theorem 1.19 for a fixed $\mu \in (0, \widehat{\mu})$.

similar approach was suggested in [41] with application to (P_λ) in the subhomogeneous case $q < p$. Unlike [27, 41], our arguments do not depend on a particular structure of the Nehari manifold of the perturbed problem, which makes them more universal.

Our last main result provides information on the nonexistence, in contrast to Theorem 1.16.

Theorem 1.23. *Let $\partial\Omega$ be C^2-smooth provided $N \geq 2$. Assume, in addition to $1 < q < p$, that*

$$(q-1)s^p + qs^{p-1} - (p-q)s + (q-p+1) \geq 0 \quad \textit{for all} \quad s \geq 0. \tag{1.10}$$

Let, moreover, one of the following assumptions be satisfied:

(i) $\int_\Omega a\varphi_p^q \, dx = 0$ *and* $\lambda > \lambda_1(p) \, (= \lambda^*)$.

(ii) $\int_\Omega a\varphi_p^q \, dx > 0$ *and* $\lambda \geq \lambda_1(p) \, (= \lambda^*)$.

Then there exists no nonnegative solution of (P_λ) *which is positive in* Ω_a^+.

The proof of Theorem 1.23 is based on the application of a generalized Picone inequality established by the present authors in [9, Theorem 1.8]. A description and some properties of the set of exponents p and q for which (1.10) holds are discussed in [9, Lemma 1.6 and Remark 1.7], and several sufficient assumptions can be also found therein. Let us explicitly emphasize two properties. First, if $p > 2q\,(> 2)$, then (1.10) is not satisfied, and hence there is no contradiction with the existence results provided by Theorem 1.16 (II), see Figure 4. Second, if $p = 2$, then (1.10) holds for all $q \in (1, 2)$, and hence Theorem 1.16 (II) cannot be extended to the case $q < p = 2$. Finally, we recall that if $\int_\Omega a\varphi_p^q\,dx \geq 0$ and $q > p$, then (P_λ) has no positive solution for $\lambda > \lambda^*$, see [25]. That is, the results of Theorem 1.16 in the case $\int_\Omega a\varphi_p^q\,dx = 0$ are of a purely nonlinear and subhomogeneous nature.

Remark 1.24. Theorem 1.23 implies that if, under the imposed assumptions, there exists a nonzero nonnegative solution u of (P_λ), then it must exhibit a dead core in Ω_a^+, that is, $u \equiv 0$ in a connected component of Ω_a^+.

Remark 1.25. We notice that the assumption (1.10) can be relaxed to $p \leq q + 1$ provided any nonzero nonnegative solution u of (P_λ) with $u > 0$ in Ω_a^+ satisfies $\nabla u \nabla \varphi_p \geq 0$ in Ω. The proof of Theorem 1.23 under these assumptions follows along the same lines, by applying [9, Theorem 1.8 (ii)].

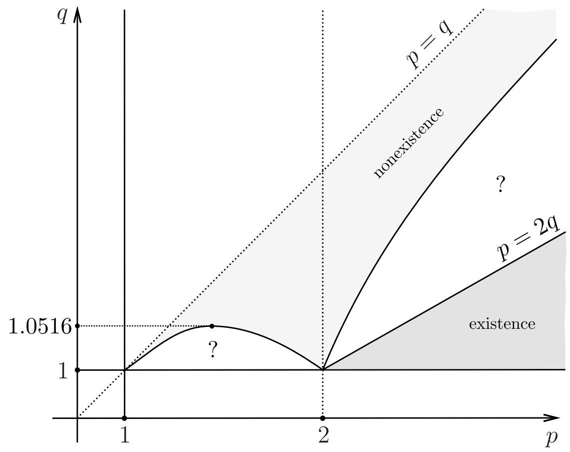

Figure 4: A schematic picture of regions in the (p, q)-plane for the validity of Theorem 1.16 (II) (gray set) and Theorem 1.23 (i) (light gray set) in the case $\int_\Omega a\varphi_p^q\,dx = 0$ and $\lambda > \lambda_1(p)$.

The rest of the work is organized as follows. In Section 2, we provide various auxiliary results on the properties of the energy functionals I_λ and $\widetilde{I}_\lambda$. Section 3 is devoted to the proof of Theorem 1.11 and to the properties of the set of nonnegative minimizers of $M(\lambda^*)$. In Section 4, we prove Propositions 1.6, 1.9, as well as several other properties of M and M^- and their minimizers. In Sections 5 and 6, we present the proofs of Theorems 1.16 and 1.19, respectively. Finally, in Section 7, we prove Theorem 1.23.

2. Auxiliary results

In this section, we collect several auxiliary results, especially on properties of the energy functionals I_λ and $\widetilde{I}_\lambda$ defined by (1.2) and (1.9), respectively, and on their critical points. Most of the results will be used in the proofs of our main theorems, while several facts also have an independent interest.

2.1. Special values of parameter

In order to provide finer analysis, we introduce the following critical values of λ in addition to λ^*, and study their relations in brief:

$$\lambda_\pm^* := \inf\left\{\frac{\|\nabla u\|_p^p}{\|u\|_p^p} :\ u \in W_0^{1,p},\ \pm\int_\Omega a|u|^q\,dx > 0\right\}, \tag{2.1}$$
$$\lambda_0^* := \inf\left\{\frac{\|\nabla u\|_p^p}{\|u\|_p^p} :\ u \in W_0^{1,p}\setminus\{0\},\ \int_\Omega a|u|^q\,dx = 0\right\}.$$

Proposition 2.1. *The following assertions hold:*

(i) *Let $\int_\Omega a\varphi_p^q\,dx > 0$. Then*
$$\lambda_1(p) = \lambda^* = \lambda_+^* < \lambda_0^* = \lambda_-^*. \tag{2.2}$$
Moreover, λ^* and λ_+^* are attained only by $t\varphi_p$ ($t \neq 0$), λ_0^* is attained, and λ_-^* is not attained.

(ii) *Let $\int_\Omega a\varphi_p^q\,dx = 0$. Then*
$$\lambda_1(p) = \lambda^* = \lambda_0^* = \lambda_\pm^*.$$
Moreover, λ^* and λ_0^* are attained only by $t\varphi_p$ ($t \neq 0$), while $\lambda_\pm^*$ is not attained.

(iii) *Let $\int_\Omega a\varphi_p^q\,dx < 0$. Then*
$$\lambda_1(p) = \lambda_-^* < \lambda_0^* = \lambda_+^* = \lambda^*.$$
Moreover, λ_-^* is attained only by $t\varphi_p$ ($t \neq 0$), λ_0^* and λ^* are attained, and λ_+^* is not attained.

Proof. We will prove only the assertion (i). The assertions (ii) and (iii) can be proved in much the same way. Clearly, the first two equalities in (2.2) and the attainability of λ^*, λ_+^* follow directly from the simplicity of $\lambda_1(p)$. The simplicity also yields $\lambda_1(p) < \lambda_0^*, \lambda_-^*$. The attainability of λ_0^* is evident.

Let us show that λ_-^* is not attained and $\lambda_0^* = \lambda_-^*$. If we suppose that λ_-^* is attained by some v, then it is attained also by $|v|$. Since $\int_\Omega a|v|^q\,dx < 0$, we see that $|v|$ is a local minimum point of the Rayleigh quotient, which means that $|v|$ is a nonnegative (consequently, positive) eigenfunction of the p-Laplacian corresponding to λ_-^*. However, the only sign-constant eigenfunction is the first one modulo scaling, see, e.g., [4, Theorem 2.1]. Hence, λ_-^* is not attained, and the corresponding minimization sequence converges to a function v satisfying $\int_\Omega a|v|^q\,dx = 0$. In particular, we deduce that $\lambda_0^* \leq \lambda_-^*$.

To complete the proof, let us obtain the converse inequality $\lambda_-^* \leq \lambda_0^*$. Let u_0 be a minimizer of λ_0^*, and hence we have $u_0 \not\equiv 0$ a.e. in Ω and $\int_\Omega a|u_0|^q\,dx = 0$. Suppose first that u_0 is not a critical point of the functional $u \mapsto \int_\Omega a|u|^q\,dx$. That is, we can find $v \in W_0^{1,p}$ such that $\int_\Omega a|u_0|^{q-2}u_0 v\,dx < 0$. So, for sufficiently small $\varepsilon > 0$ we get

$$\int_\Omega a|u_0+\varepsilon v|^q\,dx = q\int_\Omega\int_0^\varepsilon a|u_0+tv|^{q-2}(u_0+tv)v\,dtdx < 0,$$

whence $u_0+\varepsilon v$ is an admissible function for λ_-^*. Letting $\varepsilon \to 0$, we obtain $\lambda_-^* \le \lambda_0^*$.

Suppose now that $\int_\Omega a|u_0|^{q-2}u_0 v\,dx = 0$ for any $v \in W_0^{1,p}$. The fundamental lemma of the calculus of variations yields $a|u_0|^{q-2}u_0 \equiv 0$ a.e. in Ω, which implies, in its turn, that $u_0 \equiv 0$ a.e. in $\Omega_a^\pm$. Recall that $\Omega_a^- \neq \emptyset$ and take any $v \in C_0^\infty(\Omega)\setminus\{0\}$ with the support in Ω_a^-. Thanks to the equality $\int_\Omega a|u_0|^q\,dx = 0$, we obtain

$$\int_\Omega a|u_0+\varepsilon v|^q\,dx = \int_\Omega a|u_0|^q\,dx + \varepsilon^q\int_\Omega a|v|^q\,dx = \varepsilon^q\int_{\Omega_a^-} a|v|^q\,dx < 0$$

for any $\varepsilon > 0$. Consequently, $u_0+\varepsilon v$ is admissible for λ_-^*, and letting $\varepsilon \to 0$ as above, we obtain the desired inequality $\lambda_-^* \le \lambda_0^*$. □

Remark 2.2. In view of the even nature of the Rayleigh quotient, the critical values λ^*, $\lambda_\pm^*$, λ_0^* can be equivalently characterized via the truncated integrals $\int_\Omega u_+^p\,dx$ and $\int_\Omega a u_+^q\,dx$.

Remark 2.3. Several sufficient assumptions guaranteeing that the minimizer of λ_0^* generates a critical point of I_λ are presented in [41, Section 3.1].

2.2. Fibered functionals

The following results on the fibered functionals associated with I_λ and $\widetilde{I}_\lambda$ are standard and can be found, e.g., in [13] in the linear case $p = 2$, or in [22, 27] in the superhomogeneous case $q > p$.

Take any $u \in W_0^{1,p}$ such that $E_\lambda(u)\cdot\int_\Omega a|u|^q\,dx > 0$. Then there exists a unique critical point $t_\lambda(u) > 0$ of the function $t \mapsto I_\lambda(tu)$ for $t > 0$. We easily see that

$$t_\lambda(u) = \left(\frac{\int_\Omega a|u|^q\,dx}{E_\lambda(u)}\right)^{\frac{1}{p-q}} = \frac{\left|\int_\Omega a|u|^q\,dx\right|^{\frac{1}{p-q}}}{|E_\lambda(u)|^{\frac{1}{p-q}}} \tag{2.3}$$

and

$$I_\lambda(t_\lambda(u)u) = -\frac{p-q}{pq}\,E_\lambda(t_\lambda(u)u) = -t_\lambda^q(u)\,\frac{p-q}{pq}\int_\Omega a|u|^q\,dx. \tag{2.4}$$

Consequently, if $E_\lambda(u) > 0$ and $\int_\Omega a|u|^q\,dx > 0$, then $t_\lambda(u)$ is the global minimum point of the function $t \mapsto I_\lambda(tu)$ for $t > 0$, and $I_\lambda(t_\lambda(u)u) < 0$. Analogously, if $E_\lambda(u) < 0$ and $\int_\Omega a|u|^q\,dx < 0$, then $t_\lambda(u)$ is the global maximum point of the function $t \mapsto I_\lambda(tu)$ for $t > 0$, and $I_\lambda(t_\lambda(u)u) > 0$.

Noting that $E_\lambda(u)$ and $\int_\Omega a|u|^q\,dx$ are p- and q-homogeneous, respectively, we deduce that

$$J_\lambda(u) := I_\lambda(t_\lambda(u)u) = -\mathrm{sign}(E_\lambda(u))\,\frac{p-q}{pq}\,\frac{\left|\int_\Omega a|u|^q\,dx\right|^{\frac{p}{p-q}}}{|E_\lambda(u)|^{\frac{q}{p-q}}}. \tag{2.5}$$

The functional J_λ is 0-homogeneous and it is called *fibered* functional. This functional will serve as a convenient tool in the study of attainability of $M(\lambda)$.

Expressions analogous to (2.3) and (2.5) hold true for the truncated functional $\widetilde{I}_\lambda$ defined by (1.9). In particular, we denote by $\widetilde{J}_\lambda$ the truncated fibered functional.

2.3. Nehari manifold

Let us provide additional information on the Nehari manifold $\mathcal{N}_\lambda$ defined by (1.3). If $E_\lambda(u) \cdot \int_\Omega a|u|^q\,dx > 0$ for some $u \in W_0^{1,p}$, then $t_\lambda(u)u \in \mathcal{N}_\lambda$, where $t_\lambda(u) > 0$ is given by (2.3). We have

$$I_\lambda(u) = -\frac{p-q}{pq}\,E_\lambda(u) = -\frac{p-q}{pq}\int_\Omega a|u|^q\,dx \quad \text{for any } u \in \mathcal{N}_\lambda, \tag{2.6}$$

cf. (2.4). Recall the definitions (1.6) and (1.8) of the sets $\mathcal{A}^+$ and $\mathcal{A}^-$, respectively:

$$\mathcal{A}^\pm := \left\{ u \in W_0^{1,p} : \ \pm\int_\Omega a|u|^q\,dx > 0 \right\}.$$

Lemma 2.4. *The following assertions hold:*

(i) $\mathcal{N}_\lambda \cap \mathcal{A}^+ \neq \emptyset$ *for any* $\lambda \in \mathbb{R}$.

(ii) $\mathcal{N}_\lambda \cap \mathcal{A}^- \neq \emptyset$ *if and only if* $\lambda > \lambda_-^*$.

Proof. (i) Since $\Omega_a^+ \neq \emptyset$, for any $x_0 \in \Omega_a^+$ there exists a ball $B(x_0)$ centered at x_0 such that $a > 0$ a.e. in $B(x_0)$. Choose any nonnegative $v \in C_0^\infty(\Omega) \setminus \{0\}$ with the support in $B(x_0)$ and set $v_n(\cdot) = n^{N/p-1}v(n(\cdot - x_0))$. Then $E_\lambda(v_n) > 0$ for any sufficiently large n, and $\int_\Omega a v_n^q\,dx > 0$. As a result, $t_\lambda(v_n)v_n \in \mathcal{N}_\lambda \cap \mathcal{A}^+$.

(ii) It follows from the definition of λ_-^* that if $\lambda \leq \lambda_-^*$, then $E_\lambda(u) \geq 0$ for any u satisfying $\int_\Omega a|u|^q\,dx < 0$, and hence $\mathcal{N}_\lambda \cap \mathcal{A}^- = \emptyset$. On the other hand, if $\lambda > \lambda_-^*$, then there exists u such that $\int_\Omega a|u|^q\,dx < 0$ and $\lambda > \|\nabla u\|_p^p/\|u\|_p^p \geq \lambda_-^*$. Consequently, we have $E_\lambda(u) < 0$, which yields $t_\lambda(u)u \in \mathcal{N}_\lambda \cap \mathcal{A}^-$. $\square$

Remark 2.5. In view of (2.5), (2.6), and the fact that $\mathcal{N}_\lambda \cap \mathcal{A}^+ \neq \emptyset$ for any λ, we see that

$$M(\lambda) = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u) = \inf_{u \in \mathcal{N}_\lambda \cap \mathcal{A}^+} I_\lambda(u) = \inf_{u \in \mathcal{N}_\lambda \cap \mathcal{A}^+} J_\lambda(u) < 0.$$

The following result can be proved by standard arguments based on the Lagrange multipliers rule (see, e.g. [45, Theorem 48.B and Corollary 48.10]).

Lemma 2.6. *Let $u \in \mathcal{N}_\lambda$ be a critical point of I_λ over $\mathcal{N}_\lambda$. Assume that $E_\lambda(u) \neq 0$ (or, equivalently, $\int_\Omega a|u|^q\,dx \neq 0$). Then u is a critical point of I_λ (over $W_0^{1,p}$).*

Remark 2.7. Analogs of Lemmas 2.4 and 2.6 are also valid in the truncated case.

2.4. Properties of I_λ and $\widetilde{I}_\lambda$

In this subsection, we provide several results on the relation between the functionals I_λ and $\widetilde{I}_\lambda$ and on properties of their critical points.

The following result, in combination with Proposition 1.3, asserts that ground states and least $\widetilde{I}_\lambda$-energy solutions of (P_λ) correspond to the same critical level whenever either $M(\lambda)$ or $\widetilde{M}(\lambda)$ is attained.

Proposition 2.8. *Let $\lambda \in \mathbb{R}$. Then $M(\lambda)$ is attained if and only if $\widetilde{M}(\lambda)$ is attained. Moreover, $M(\lambda) = \widetilde{M}(\lambda) < 0$ for any $\lambda \in \mathbb{R}$.*

Proof. Observe that $\widetilde{M}(\lambda) \leq M(\lambda)$ for any $\lambda \in \mathbb{R}$. Indeed, if $u \in \mathcal{N}_\lambda$, then $|u| \in \mathcal{N}_\lambda$ and hence $|u| \in \widetilde{\mathcal{N}}_\lambda$. Since $\widetilde{I}_\lambda(|u|) = I_\lambda(|u|) = I_\lambda(u)$, the inequality $\widetilde{M}(\lambda) \leq M(\lambda)$ follows. As a consequence, we have $\widetilde{M}(\lambda) = M(\lambda) = -\infty$ for any $\lambda > \lambda^*$, see Theorem 1.5 (ii). Moreover, $\widetilde{M}(\lambda), M(\lambda) < 0$ for any $\lambda \in \mathbb{R}$, see Remark 2.5.

Suppose, by contradiction, that $\widetilde{M}(\lambda) < M(\lambda)$ for some $\lambda \leq \lambda^*$. That is, there exists $u \in \widetilde{\mathcal{N}}_\lambda$ such that $\widetilde{I}_\lambda(u) < M(\lambda) < 0$. Clearly, we have $u_+ \not\equiv 0$ a.e. in Ω. We also conclude that $u_- \not\equiv 0$ a.e. in Ω, since in the case $u \geq 0$ a.e. in Ω we would get $u \in \mathcal{N}_\lambda$ and $\widetilde{I}_\lambda(u) = I_\lambda(u) \geq M(\lambda)$. The assumptions on u yield

$$\widetilde{E}_\lambda(u) = \int_\Omega |\nabla u_-|^p \, dx + E_\lambda(u_+) = \int_\Omega a u_+^q \, dx > 0. \tag{2.7}$$

If $E_\lambda(u_+) > 0$, then $t_\lambda(u_+)u_+ \in \mathcal{N}_\lambda$ and

$$I_\lambda(u_+) \geq \min_{t>0} I_\lambda(tu_+) = I_\lambda(t_\lambda(u_+)u_+) \geq M(\lambda),$$

which is impossible since

$$M(\lambda) > \widetilde{I}_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u_-|^p \, dx + I_\lambda(u_+) > I_\lambda(u_+).$$

Therefore, we have $E_\lambda(u_+) \leq 0$ and hence, recalling that $u_+ \not\equiv 0$ a.e. in Ω, we obtain

$$\frac{\int_\Omega |\nabla u_+|^p \, dx}{\int_\Omega u_+^p \, dx} \leq \lambda \leq \lambda^* = \lambda_+^* \leq \frac{\int_\Omega |\nabla u_+|^p \, dx}{\int_\Omega u_+^p \, dx}. \tag{2.8}$$

Here, the equality is given by Proposition 2.1, and the last inequality follows from the definition (2.1) of λ_+^* and the fact that $\int_\Omega a u_+^q \, dx > 0$, see (2.7). We conclude from (2.8) that u_+ is a minimizer of λ_+^*. However, according to Proposition 2.1, λ_+^* is either not attained, or attained by $t\varphi_p$ with some $t \neq 0$, which contradicts the fact that $u_- \not\equiv 0$ a.e. in Ω. $\square$

Remark 2.9. In general, $M(\lambda)$ might possess sign-changing minimizers (since the strong maximum principle is not applicable to nonnegative solutions of (P_λ)), while the set of minimizers of $\widetilde{M}(\lambda)$ consists solely of nonnegative functions. That is, the set of minimizers of $\widetilde{M}(\lambda)$ is contained in that of $M(\lambda)$.

Remark 2.10. Let u be a *nonnegative* solution of (P_λ). If u has a zero point x_0 in Ω_a^+, then, according to the strong maximum principle (see, e.g., [40, Theorem 1.1]), $u \equiv 0$ in a connected component of Ω_a^+ which contains x_0. In other words, if $u \not\equiv 0$ in some connected component of Ω_a^+, then $u > 0$ in that connected component. An additional assumption on u guaranteeing that $u > 0$ in the whole open set Ω_a^+ is presented in Lemma 2.11.

The proofs of all remaining results of this subsection will be given only for the untruncated functional I_λ. The case of $\widetilde{I}_\lambda$ can be treated analogously.

Lemma 2.11. *Let u be a local minimum point of I_λ or of $\widetilde{I}_\lambda$ which is nonnegative in Ω_a^+. Then $u > 0$ in Ω_a^+.*

Proof. Suppose, contrary to our claim, that $u(x_0) = 0$ for some $x_0 \in \Omega_a^+$. By the strong maximum principle, we have $u \equiv 0$ in a connected component A of Ω_a^+ containing x_0. Consider any nonnegative $\varphi \in C_0^\infty(\Omega) \setminus \{0\}$ with the support in A. Since $q < p$ and $\int_\Omega a\varphi^q \, dx > 0$, we have $I_\lambda(t\varphi) < 0$ for any sufficiently small $t > 0$. This yields

$$I_\lambda(u) \leq I_\lambda(u + t\varphi) = I_\lambda(u) + I_\lambda(t\varphi) < I_\lambda(u),$$

which contradicts our assumption that u is a local minimum point of I_λ. $\square$

Remark 2.12. One could wonder whether it is possible to rid of the nonnegativity assumption in Lemma 2.11. In general, this assumption is vital. In [11, Section 5], it is shown that there are domains for which a least energy *nodal* solution to the zero Dirichlet problem for the equation $-\Delta u = |u|^{q-2}u$, i.e., (P_λ) with $a \equiv 1$ and $\lambda = 0$, is a point of local minimum.

Remark 2.13. It would be interesting to know whether the result of Lemma 2.11 remains valid for ground states or least $\widetilde{I}_\lambda$-energy solutions of (P_λ), cf. Lemma 2.16 below.

Another general results are given in the following two lemmas.

Lemma 2.14. *Let u be a local minimum point of I_λ (resp. $\widetilde{I}_\lambda$). Then $I_\lambda(u) < 0$ (resp. $\widetilde{I}_\lambda(u) < 0$).*

Proof. Observe that u is a solution of (P_λ), and hence $u \in \mathcal{N}_\lambda$ and $t_\lambda(u) = 1$. Suppose first that $I_\lambda(u) > 0$. By (2.4), $t_\lambda(u)$ is the global maximum point of the function $t \mapsto I_\lambda(tu)$, $t > 0$, and it is clear that $I_\lambda(u) > I_\lambda(tu)$ for any $t \neq 1$. This contradicts our assumption that u is a local minimizer of I_λ.

Suppose now that $I_\lambda(u) = 0$. We get from (2.6) that $E_\lambda(u) = 0 = \int_\Omega a|u|^q \, dx$. Clearly, $u \not\equiv 0$ in Ω since 0 is not a local minimum point of I_λ in view of the assumption $q < p$ and the fact that $\mathcal{A}^+ \neq \emptyset$ (see also Lemma 2.11). Moreover, u is not a critical point of the functional $w \mapsto \int_\Omega a|w|^q \, dx$. Indeed, if u is such a critical point, then $\int_\Omega a|u|^{q-2}uv \, dx = 0$ for all $v \in W_0^{1,p}$. This yields $a|u|^{q-2}u \equiv 0$ a.e. in Ω and hence $u \equiv 0$ in Ω_a^+. However, Lemma 2.11 implies that $u > 0$ in Ω_a^+, which is a contradiction. Thus, we can find $v \in W_0^{1,p}$ such that $\int_\Omega a|u|^{q-2}uv \, dx < 0$. Since u is a solution of (P_λ), we have $\frac{1}{p}\langle E_\lambda'(u), v\rangle = \int_\Omega a|u|^{q-2}uv \, dx < 0$. Therefore, we see that tu with $t > 1$ is not a critical point of I_λ:

$$\begin{aligned}
\langle I_\lambda'(tu), v\rangle &= t^{q-1}\left(\frac{t^{p-q}}{p}\langle E_\lambda'(u), v\rangle - \int_\Omega a|u|^{q-2}uv \, dx\right) \\
&= t^{q-1}\left(t^{p-q} - 1\right)\int_\Omega a|u|^{q-2}uv \, dx < 0.
\end{aligned}$$

At the same time, we have $I_\lambda(tu) = 0$ for any $t > 0$ since $E_\lambda(u) = 0 = \int_\Omega a|u|^q \, dx$. Thus, by the mean value theorem, for any $t > 1$ and for any sufficiently small $s > 0$ there exists $s_0 \in (0, s)$ such that

$$I_\lambda(tu + sv) = I_\lambda(tu) + s\langle I_\lambda'(tu + s_0 v), v\rangle = s\langle I_\lambda'(tu + s_0 v), v\rangle < 0.$$

Consequently, taking $(t-1)$ and s small enough, we conclude that $tu + sv$ belongs to a small neighborhood of the local minimum point u, but $I_\lambda(tu+sv) < 0 = I_\lambda(u)$. A contradiction. $\square$

Remark 2.15. Notice that in Lemmas 2.11 and 2.14 the local minimum point u of I_λ is *not* required to be of constant sign in Ω.

Lemma 2.16. *Let u be a nonzero critical point of I_λ (resp. of $\widetilde{I}_\lambda$) such that $u \equiv 0$ in Ω_a^+. Then $I_\lambda(u) \geq 0$ (resp. $\widetilde{I}_\lambda(u) \geq 0$). Let, in addition, at least one of the following assumptions be satisfied:*

(i) *$\Omega \setminus (\Omega_a^+ \cup \Omega_a^-)$ has empty interior.*

(ii) *u has a constant sign in Ω.*

(iii) *$p = 2$.*

(iv) *$N = 1$.*

Then $I_\lambda(u) > 0$ (resp. $\widetilde{I}_\lambda(u) > 0$).

Proof. Since $u \equiv 0$ in Ω_a^+, we have $\int_\Omega a|u|^q\,dx \leq 0$ in view of the third assumptions in (1.1). Thus, $I_\lambda(u) \geq 0$ by (2.6). In order to prove the second part of the lemma, let us exclude the case $I_\lambda(u) = 0$. Suppose that $I_\lambda(u) = 0$, and hence $\int_\Omega a|u|^q\,dx = 0$. Recalling that $u \in C_0^{1,\beta}(\Omega)$ by Remark 1.1, we see that for any $x_0 \in \Omega$ satisfying $u(x_0) \neq 0$ there exists an open ball $B(x_0)$ centered at x_0 such that $a \equiv 0$ a.e. in $B(x_0)$. Thus, $B(x_0) \subset \Omega \setminus (\Omega_a^+ \cup \Omega_a^-)$. We conclude that $u \equiv 0$ in Ω under the assumption (i), which is impossible. In general, we see that u weakly solves the equation $-\Delta_p u = \lambda|u|^{p-2}u$ in Ω, and u is zero in the open subset $\Omega_a^+ \cup \Omega_a^-$ of Ω. Under the assumption (ii), we have $u \equiv 0$ in Ω thanks to the strong maximum principle, a contradiction. If the assumption (iii) holds, then u obeys the unique continuation property (see, e.g., [37]) and hence u cannot vanish in an open subset of Ω. Finally, under the assumption (iv), all eigenfunctions of the p-Laplacian have explicit structure and cannot vanish in open intervals as well, see, e.g., [21]. $\square$

Remark 2.17. The proof of Lemma 2.16 in the case $p = 2$ relies of the unique continuation property (UCP). Assuming the UCP is true for some $p \neq 2$, the statement of Lemma 2.16 can be generalized accordingly. However, to the best of our knowledge, the validity of the UCP is not known for sign-changing eigenfunctions of the p-Laplacian in higher dimensions $N \geq 2$.

The following result complements [32, Proposition 3.9 (1)].

Proposition 2.18. *Assume that $\lambda \leq \lambda_-^*$. Then, I_λ and $\widetilde{I}_\lambda$ have no positive critical values.*

Proof. Let u be a nonzero critical point of I_λ, and hence $u \in \mathcal{N}_\lambda$. Since $\mathcal{N}_\lambda \cap \mathcal{A}^- = \emptyset$ for $\lambda \leq \lambda_-^*$ (see Lemma 2.4), we have $\int_\Omega a|u|^p\,dx \geq 0$, which yields $I_\lambda(u) \leq 0$ by (2.6). If u is a nonzero critical point of $\widetilde{I}_\lambda$, then u is a nonzero nonnegative solution of (P_λ) (see Section 1.2). That is, u is a nonzero critical point of I_λ, and the conclusion follows as above. $\square$

In contrast to Proposition 2.18, the functionals I_λ and $\widetilde{I}_\lambda$ might possess positive critical values for sufficiently large λ.

Proposition 2.19. *Assume that $a \equiv const < 0$ in some open ball $B \subset \Omega_a^-$. Then there exists a sufficiently large $\bar{\lambda}$ such that (P_λ) possesses a nonnegative solution with compact support in B and positive energy for any $\lambda > \bar{\lambda}$.*

Proof. Consider the Dirichlet problem

$$\begin{cases} -\Delta_p u = |u|^{p-2}u - |u|^{q-2}u & \text{in } B_R, \\ \quad u = 0 & \text{on } \partial B_R, \end{cases} \tag{2.9}$$

where $B_R \subset \mathbb{R}^N$ is an open ball of some radius $R > 0$ centered at the origin. It is known from, e.g., [17, Theorem 1] (for $N = 1$) and [24, Proposition 2] (for $N \geq 2$) that there exists $R > 0$ such that (2.9) has a radially symmetric nonnegative solution u with compact support in B_R. Moreover, we have $u \in C_0^1(\overline{B_R})$, see Remark 1.1. By considering a function $v \in C_0^1(\overline{B})$ defined as $v(x) = Au(C(x - x_0))$ with appropriate constants $A, C, x_0 > 0$, it can be derived in much the same way as in [18, Proposition 5.1] that v is a compact support solution of (P_λ) in the ball B (given in the statement of the proposition) for any sufficiently large λ. Extending v by zero outside of B, we obtain a nonnegative solution of (P_λ) in Ω (or even in the whole $\mathbb{R}^N$). Applying Lemma 2.16 (ii), we conclude that v has positive energy. $\square$

Remark 2.20. The assumption of Proposition 2.19 on the weight a can be weakened to cover some nonconstant weights, see, e.g., [18, Remark 5.2 and Theorem 5.1] for the linear case $p = 2$.

2.5. Palais–Smale type conditions

We provide two standard but useful compactness results. We denote by $\|\cdot\|_*$ the usual operator norm.

Lemma 2.21. *Let $\{\lambda_n\} \subset \mathbb{R}$ converge to some $\lambda \in \mathbb{R}$. Let $\{u_n\}$ be a bounded sequence in $W_0^{1,p}$ such that $\|I'_{\lambda_n}(u_n)\|_* \to 0$ as $n \to +\infty$. Then $\{u_n\}$ has a subsequence strongly convergent in $W_0^{1,p}$ to a solution of (P_λ).*

Proof. Since $\{u_n\}$ is bounded, we may assume that it converges to some $u_0 \in W_0^{1,p}$ weakly in $W_0^{1,p}$ and strongly in $L^p(\Omega)$, up to a subsequence. We obtain from the convergence $\|I'_{\lambda_n}(u_n)\|_* \to 0$ that

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u_0)\, dx \\ &\qquad = \lambda_n \int_\Omega |u_n|^{p-2} u_n (u_n - u_0)\, dx + \int_\Omega a|u_n|^{q-2} u_n (u_n - u_0)\, dx + o(1) \to 0 \end{aligned}$$

as $n \to +\infty$. In view of the (S_+)-property of the p-Laplacian (see, e.g., [20, Theorem 10]), $u_n \to u_0$ strongly in $W_0^{1,p}$. As a consequence, we easily conclude that u_0 is a solution of (P_λ). $\square$

Lemma 2.22. *Let $\lambda \neq \lambda_1(p)$. Then any sequence $\{u_n\} \subset W_0^{1,p}$ such that $\|\widetilde{I}'_\lambda(u_n)\|_* \to 0$ as $n \to +\infty$ has a subsequence strongly convergent in $W_0^{1,p}$ to a nonnegative solution of (P_λ). In particular, $\widetilde{I}_\lambda$ satisfies the Palais–Smale condition.*

Proof. Let $\{u_n\}$ satisfy $\|\widetilde{I}'_\lambda(u_n)\|_* \to 0$ as $n \to +\infty$. Due to the (S_+)-property of the p-Laplacian (see, e.g., [20, Theorem 10]), $\{u_n\}$ has a subsequence strongly convergent in $W_0^{1,p}$ provided $\{u_n\}$ is bounded. Suppose, contrary to our claim, that $\|\nabla u_n\|_p \to +\infty$, up to a subsequence, and consider the normalized functions $v_n := u_n/\|\nabla u_n\|_p$. Let us show that $v_n \to \varphi_p$ strongly in $W_0^{1,p}$. Since $\{v_n\}$ is bounded, it converges to some $v_0 \in W_0^{1,p}$ weakly in $W_0^{1,p}$ and strongly in $L^p(\Omega)$, up to a subsequence. We have

$$\left|\frac{1}{p}\langle \widetilde{E}_\lambda'(v_n), \xi\rangle - \frac{1}{\|\nabla u_n\|_p^{p-q}} \int_\Omega a(v_n)_+^{q-1}\xi\, dx\right| = \frac{|\langle \widetilde{I}_\lambda'(u_n), \xi\rangle|}{\|\nabla u_n\|_p^{p-1}} \le \frac{\|\nabla \xi\|_p \|\widetilde{I}_\lambda'(u_n)\|_*}{\|\nabla u_n\|_p^{p-1}} \tag{2.10}$$

for any $\xi \in W_0^{1,p}$. This yields $\langle \widetilde{E}_\lambda'(v_0), \xi\rangle = 0$ for any $\xi \in W_0^{1,p}$. Moreover, taking $\xi = v_n$ in (2.10), we get $\lim\limits_{n\to+\infty} \langle \widetilde{E}_\lambda'(v_n), v_n\rangle = 0$ and hence $v_0 \not\equiv 0$ a.e. in Ω. That is, v_0 is an eigenfunction of the p-Laplacian corresponding to λ. Noting that

$$\|\nabla (u_n)_-\|_p^p = |\langle \widetilde{I}_\lambda'(u_n), (u_n)_-\rangle| \le \|\widetilde{I}_\lambda'(u_n)\|_* \|\nabla (u_n)_-\|_p,$$

we conclude that $v_0 \ge 0$ in Ω. Therefore, $\lambda = \lambda_1(p)$ and $v_0 = \varphi_p$, since φ_p is the only eigenfunction of the p-Laplacian with constant sign in Ω (see, e.g., [4]). This contradicts our assumption $\lambda \neq \lambda_1(p)$. □

Finally, we show that Lemma 2.22 leads to the following existence result.

Proposition 2.23. *Let $\lambda \neq \lambda_1(p)$. Assume that $\widetilde{I}_\lambda$ has a critical point u satisfying $\widetilde{I}_\lambda(u) < 0$. Then there exists a least $\widetilde{I}_\lambda$-energy solution w of (P_λ) and $w > 0$ in some connected component of Ω_a^+.*

Proof. Denote by $\mathscr{S}_\lambda$ the set of all critical points of $\widetilde{I}_\lambda$ and define $d := \inf\{\widetilde{I}_\lambda(v) : v \in \mathscr{S}_\lambda\}$. Let $\{w_n\} \subset \mathscr{S}_\lambda$ be a minimizing sequence for d. Since $\widetilde{I}_\lambda'(w_n) = 0$ for all n, $\{w_n\}$ has a subsequence strongly convergent in $W_0^{1,p}$ to some nonnegative solution w of (P_λ), thanks to Lemma 2.22. Clearly, we have $\widetilde{I}_\lambda(w) = d$, whence w is a least $\widetilde{I}_\lambda$-energy solution of (P_λ). Thanks to the assumption $\widetilde{I}_\lambda(u) < 0$, we have $\widetilde{I}_\lambda(w) < 0$, and hence w is nonzero. Applying Lemma 2.16 and noting Remark 2.10, we deduce that $w > 0$ at least in some connected component of Ω_a^+. □

2.6. Behavior of sequences

In this subsection, we collect several results on the behavior of functional sequences, which will be used in the proofs of our main theorems.

Lemma 2.24. *Let $\{\lambda_n\} \subset \mathbb{R}$ and $\{u_n\} \subset W_0^{1,p}$ be sequences such that $u_n \ge 0$ a.e. in Ω for all $n \in \mathbb{N}$ and*

$$\lambda_n \to \lambda \in \mathbb{R}, \quad \|\nabla u_n\|_p \to +\infty, \quad \|I_{\lambda_n}'(u_n)\|_* \to 0 \quad \text{as } n \to +\infty.$$

Then $\lambda = \lambda_1(p)$ and the sequence $\{v_n\}$, where $v_n := u_n/\|\nabla u_n\|_p$, has a subsequence strongly convergent in $W_0^{1,p}$ to φ_p.

Proof. Since $\{v_n\}$ is bounded in $W_0^{1,p}$, we may suppose that $\{v_n\}$ converges to some $v_0 \in W_0^{1,p}$

weakly in $W_0^{1,p}$ and strongly in $L^p(\Omega)$ and $L^q(\Omega)$, up to a subsequence. Consequently, we get

$$\begin{aligned}
o(1) &= \left\langle I'_{\lambda_n}(u_n), \frac{v_n - v_0}{\|\nabla u_n\|_p^{p-1}} \right\rangle \\
&= \int_\Omega |\nabla v_n|^{p-2} \nabla v_n \nabla (v_n - v_0)\, dx \\
&\quad - \lambda_n \int_\Omega |v_n|^{p-2} v_n (v_n - v_0)\, dx - \frac{1}{\|\nabla u_n\|_p^{p-q}} \int_\Omega a |v_n|^{q-2} v_n (v_n - v_0)\, dx \\
&= \int_\Omega |\nabla v_n|^{p-1} \nabla v_n \nabla (v_n - v_0)\, dx + o(1).
\end{aligned}$$

Hence, the (S_+)-property of the p-Laplacian implies that $v_n \to v_0$ strongly in $W_0^{1,p}$ (see, e.g., [20, Theorem 10]), whence $\|\nabla v_0\|_p = 1$ and $v_0 \not\equiv 0$ a.e. in Ω.

Considering now $\langle I'_{\lambda_n}(u_n), \xi / \|\nabla u_n\|_p^{p-1} \rangle$ for any $\xi \in W_0^{1,p}$, we have

$$\int_\Omega |\nabla v_n|^{p-2} \nabla v_n \nabla \xi\, dx - \lambda_n \int_\Omega |v_n|^{p-2} v_n \xi\, dx - \frac{1}{\|\nabla u_n\|_p^{p-q}} \int_\Omega a |v_n|^{q-2} v_n \xi\, dx = o(1).$$

Letting $n \to +\infty$ and recalling that $\|\nabla v_0\|_p = 1$ and $v_n \geq 0$ a.e. in Ω for all n, we deduce that v_0 is a nonnegative eigenfunction of the p-Laplacian. This yields $\lambda = \lambda_1(p)$, since any higher eigenfunction must be sign-changing (see, e.g., [4, Theorem 2.1]). Finally, the simplicity of $\lambda_1(p)$ ensures that $v_0 = \varphi_p$. $\square$

Lemma 2.25. *Let $\{\lambda_n\} \subset \mathbb{R}$ be such that $M(\lambda_n)$ is attained, and $\lambda_n \to \lambda \in \mathbb{R}$ as $n \to +\infty$. Let u_n be a minimizer of $M(\lambda_n)$, and let $\{u_n\}$ be bounded in $W_0^{1,p}$. If $\liminf\limits_{n\to+\infty} I_{\lambda_n}(u_n) < 0$, then $\{u_n\}$ has a subsequence strongly convergent in $W_0^{1,p}$ to a minimizer of $M(\lambda)$.*

Proof. Applying Lemma 2.21, we see that $\{u_n\}$ converges to a solution u_0 of (P_λ) strongly in $W_0^{1,p}$, up to a subsequence. The assumption $\liminf\limits_{n\to+\infty} I_{\lambda_n}(u_n) < 0$ guarantees that $u_0 \not\equiv 0$ in Ω, and hence $u_0 \in \mathcal{N}_\lambda$. Let us prove that u_0 is a minimizer of $M(\lambda)$. Fix any $w \in \mathcal{N}_\lambda \cap \mathcal{A}^+$. By the continuity, the strict inequality $E_{\lambda_n}(w) > 0$ holds for all sufficiently large n, and we obtain

$$I_{\lambda_n}(u_n) = M(\lambda_n) \leq I_{\lambda_n}(t_{\lambda_n}(w)w) = \min_{t>0} I_{\lambda_n}(tw) \leq I_{\lambda_n}(w).$$

Letting $n \to +\infty$, we arrive at $I_\lambda(u_0) \leq I_\lambda(w)$, and therefore u_0 is a minimizer of $M(\lambda)$, see Remark 2.5. $\square$

The following auxiliary result will be essential to prove Theorem 1.11 (ii) and Lemma 3.1.

Lemma 2.26. *Let $p \geq 2q$. Assume $\partial\Omega$ to be C^2-smooth and connected provided $N \geq 2$. Let $\lambda = \lambda_1(p)$ and $\int_\Omega a \varphi_p^q\, dx = 0$. Let $\{w_n\} \subset W_0^{1,p}$ be such that $E_\lambda(w_n) > 0$, $\int_\Omega a|w_n|^q\, dx > 0$ for any $n \in \mathbb{N}$, and $w_n \to \varphi_p$ strongly in $W_0^{1,p}$. Then the following assertions hold:*

(i) *If $p = 2q$, then*

$$-\infty < \liminf_{n\to+\infty} I_\lambda(t_\lambda(w_n)w_n) \leq \limsup_{n\to+\infty} I_\lambda(t_\lambda(w_n)w_n) \leq 0.$$

(ii) *If $p > 2q$, then* $\lim_{n\to+\infty} I_\lambda(t_\lambda(w_n)w_n) = 0$.

Proof. Throughout the proof, we always denote by C a positive constant independent of $n \in \mathbb{N}$. We decompose w_n as $w_n = \gamma_n\varphi_p + v_n$, where $\gamma_n \in \mathbb{R}$ and $v_n \in W_0^{1,p}$ are chosen in such a way that $\gamma_n = \|\varphi_p\|_2^{-2}\int_\Omega w_n\varphi_p\,dx$ and $\int_\Omega v_n\varphi_p\,dx = 0$ for all n. Notice that $v_n \neq 0$ for all n because $E_\lambda(w_n) > 0 = E_\lambda(\varphi_p)$. Since $w_n \to \varphi_p$ strongly in $W_0^{1,p}$, it is not hard to deduce that

$$\gamma_n \to 1 \quad \text{and} \quad \|\nabla v_n\|_p \to 0 \quad \text{as} \quad n \to +\infty. \tag{2.11}$$

Recalling now that $\partial\Omega$ is C^2-smooth and connected when $N \geq 2$, we can use the improved Poincaré inequality from [23] (see [23, Corollary 1.2]) to provide the lower bound

$$E_\lambda(w_n) \geq C\left(|\gamma_n|^{p-2}\int_\Omega |v_n|^2\,dx + \int_\Omega |v_n|^p\,dx\right) \geq \frac{C}{2}\left(\int_\Omega |v_n|^2\,dx + \int_\Omega |v_n|^p\,dx\right) > 0 \tag{2.12}$$

for all sufficiently large n. Let us now estimate $\int_\Omega a|w_n|^q\,dx$ from above. Recalling that $\int_\Omega a\varphi_p^q\,dx = 0$ and applying the mean value theorem, for each n we can find $\varepsilon_n \in (0,1)$ such that

$$\begin{aligned} 0 < \int_\Omega a|w_n|^q\,dx &= |\gamma_n|^q\int_\Omega a\varphi_p^q\,dx + q\int_\Omega a|\gamma_n\varphi_p + \varepsilon_n v_n|^{q-2}(\gamma_n\varphi_p + \varepsilon_n v_n)v_n\,dx \\ &\leq q\int_\Omega |a||\gamma_n\varphi_p + \varepsilon_n v_n|^{q-1}|v_n|\,dx. \end{aligned}$$

Since $p \geq 2q > 2(q-1)$ by our assumption, we use the Hölder inequality and the convergences (2.11) to obtain the following upper bound for all n:

$$\begin{aligned} q\int_\Omega |a||\gamma_n\varphi_p + \varepsilon_n v_n|^{q-1}|v_n|\,dx &\leq C\left(\int_\Omega |\gamma_n\varphi_p + \varepsilon_n v_n|^{2(q-1)}\,dx\right)^{\frac{1}{2}}\left(\int_\Omega |v_n|^2\,dx\right)^{\frac{1}{2}} \\ &\leq C\left(\int_\Omega |v_n|^2\,dx\right)^{\frac{1}{2}} \leq C\left(\int_\Omega |v_n|^2\,dx + \int_\Omega |v_n|^p\,dx\right)^{\frac{1}{2}}. \end{aligned} \tag{2.13}$$

Combining now (2.12) and (2.13), and recalling that $p \geq 2q$, we deduce that

$$\begin{aligned} 0 &\geq \limsup_{n\to+\infty} I_\lambda(t_\lambda(w_n)w_n) \geq \liminf_{n\to+\infty} I_\lambda(t_\lambda(w_n)w_n) \\ &= \liminf_{n\to+\infty} J_\lambda(w_n) \geq -C\limsup_{n\to+\infty}\left(\int_\Omega |v_n|^2\,dx + \int_\Omega |v_n|^p\,dx\right)^{\frac{p-2q}{2(p-q)}} > -\infty. \end{aligned}$$

Moreover, if $p > 2q$, then we see that $\lim_{n\to+\infty} I_\lambda(t_\lambda(w_n)w_n) = 0$. □

3. The least energy at λ^*

In this section, we prove Theorem 1.11 and provide several auxiliary results on the properties of the critical set of $M(\lambda^*)$ which will be crucial for the proof of Theorem 1.16 given in Section 5 below.

3.1. Proof of Theorem 1.11

Proof of the assertion (i). As was mentioned in Remark 1.12, the proof can be found in [41]. We provide alternative arguments for the sake of completeness and clarity. Let us choose an increasing sequence $\{\lambda_n\} \subset \mathbb{R}$ converging to λ^*, and for each λ_n we denote by u_n a nonnegative ground state of (P_{λ_n}), see Theorem 1.5 for the existence. Lemma 2.24 ensures the boundedness of $\{u_n\}$ in $W_0^{1,p}$ because $\lambda^* > \lambda_1(p)$ by Proposition 2.1. Let us prove that $\liminf_{n\to+\infty} I_{\lambda_n}(u_n) < 0$. If this claim is established, then Lemma 2.25 implies the assertion (i). Fix any $w \in \mathcal{N}_{\lambda^*} \cap \mathcal{A}^+$. Since $E_{\lambda_n}(w) > 0$ for any sufficiently large n, we get $t_{\lambda_n}(w)w \in \mathcal{N}_{\lambda_n} \cap \mathcal{A}^+$, and hence

$$I_{\lambda_n}(u_n) = M(\lambda_n) \le I_{\lambda_n}(t_{\lambda_n}(w)w) = \min_{t>0} I_{\lambda_n}(tw) \le I_{\lambda_n}(w). \tag{3.1}$$

Recalling that $w \in \mathcal{N}_{\lambda^*} \cap \mathcal{A}^+$ and passing to the limit in (3.1), we arrive at

$$\limsup_{n\to+\infty} I_{\lambda_n}(u_n) = \limsup_{n\to+\infty} M(\lambda_n) \le I_{\lambda^*}(w) < 0.$$

Applying Lemma 2.25, we finish the proof. $\square$

Proof of the assertion (ii). Assume first that $p < 2q$. We have $\lambda^* = \lambda_1(p)$ by Proposition 2.1, and so $E_{\lambda^*}(\varphi_p) = 0$ and

$$\langle E'_{\lambda^*}(\varphi_p), \theta \rangle = 0 \quad \text{for any} \quad \theta \in C_0^\infty(\Omega). \tag{3.2}$$

Since $\varphi_p > 0$ in Ω, we can find $\theta \in C_0^\infty(\Omega)$ satisfying $\int_\Omega a\varphi_p^{q-1}\theta \, dx > 0$. By the continuity, there exist $\varepsilon_0 > 0$ and $C > 0$ such that

$$q \int_\Omega a|\varphi_p + \varepsilon\theta|^{q-2}(\varphi_p + \varepsilon\theta)\theta \, dx \ge C > 0 \quad \text{for all } \varepsilon \in [-\varepsilon_0, \varepsilon_0]. \tag{3.3}$$

Notice that $\theta \notin \mathbb{R}\varphi_p$ in view of the assumption $\int_\Omega a\varphi_p^q \, dx = 0$. Therefore, the simplicity of $\lambda_1(p)$ gives $E_{\lambda^*}(\varphi_p + \varepsilon\theta) > 0$ for any $\varepsilon \neq 0$.

Fix any $\varepsilon \in (0, \varepsilon_0]$ and denote $u_\varepsilon := \varphi_p + \varepsilon\theta$. According to the mean value theorem, there exist $\varepsilon_1 \in (0, \varepsilon)$ and $\varepsilon_2 \in (0, \varepsilon)$ such that

$$0 < E_{\lambda^*}(u_\varepsilon) = E_{\lambda^*}(\varphi_p) + \varepsilon\langle E'_{\lambda^*}(\varphi_p + \varepsilon_1\theta), \theta\rangle = \varepsilon\langle E'_{\lambda^*}(\varphi_p + \varepsilon_1\theta), \theta\rangle \tag{3.4}$$

and, by (3.3),

$$\int_\Omega a|u_\varepsilon|^q \, dx = \int_\Omega a\varphi_p^q \, dx + \varepsilon q \int_\Omega a|\varphi_p + \varepsilon_2\theta|^{q-2}(\varphi_p + \varepsilon_2\theta)\theta \, dx \ge \varepsilon C > 0. \tag{3.5}$$

Consequently, we have $t_{\lambda^*}(u_\varepsilon)u_\varepsilon \in \mathcal{N}_{\lambda^*} \cap \mathcal{A}^+$. Our aim now is to study the behavior of $I_{\lambda^*}(t_{\lambda^*}(u_\varepsilon)u_\varepsilon)$ as $\varepsilon \to 0$. To this end, we estimate the right-hand side of (3.4) from above. It is known that there exists $C_1 > 0$ such that

$$0 \le \langle |x|^{p-2}x - |y|^{p-2}y, x - y\rangle_{\mathbb{R}^N} \le \begin{cases} C_1|x-y|^p & \text{if } 1 < p \le 2, \\ C_1|x-y|^2(|x|+|y|)^{p-2} & \text{if } p \ge 2, \end{cases} \tag{3.6}$$

for all $x, y \in \mathbb{R}^N$, see e.g., [34, Chapter 12, Eq. (III)] for $1 < p \le 2$ and [14, Proposition 17.3] for $p \ge 2$. Recalling $0 < \varepsilon_1 < \varepsilon \le \varepsilon_0$ and applying (3.2) and (3.6), we obtain

$$\begin{aligned}
\langle E'_{\lambda^*}(\varphi_p + \varepsilon_1\theta), \theta\rangle &= \langle E'_{\lambda^*}(\varphi_p + \varepsilon_1\theta), \theta\rangle - \langle E'_{\lambda^*}(\varphi_p), \theta\rangle \\
&= \frac{1}{\varepsilon_1}\langle E'_{\lambda^*}(\varphi_p + \varepsilon_1\theta) - E'_{\lambda^*}(\varphi_p), (\varphi_p + \varepsilon_1\theta) - \varphi_p\rangle \\
&\le \begin{cases} \dfrac{C_1\varepsilon_1^p}{\varepsilon_1}\|\nabla\theta\|_p^p \le C_2\varepsilon_1^{p-1} & \text{if } 1 < p \le 2, \\ \dfrac{C_1\varepsilon_1^2}{\varepsilon_1}\displaystyle\int_\Omega |\nabla\theta|^2(2|\nabla\varphi_p| + \varepsilon_1|\nabla\theta|)^{p-2}\,dx \le C_2\varepsilon_1 & \text{if } p \ge 2, \end{cases}
\end{aligned}$$

where $C_2 > 0$ does not depend on ε_1. Thus, we conclude from (3.4) that

$$E_{\lambda^*}(u_\varepsilon) \le \begin{cases} C_2\varepsilon^p & \text{if } 1 < p \le 2, \\ C_2\varepsilon^2 & \text{if } p \ge 2. \end{cases} \tag{3.7}$$

Recalling that $t_{\lambda^*}(u_\varepsilon)u_\varepsilon \in \mathcal{N}_\lambda \cap \mathcal{A}^+$ and using the estimates (3.5) and (3.7), we arrive at

$$\begin{aligned}
M(\lambda^*) &\le I_{\lambda^*}(t_{\lambda^*}(u_\varepsilon)u_\varepsilon) \\
&= J_{\lambda^*}(u_\varepsilon) = -\frac{p-q}{pq}\,\frac{\left(\int_\Omega a|u_\varepsilon|^q\,dx\right)^{\frac{p}{p-q}}}{(E_{\lambda^*}(u_\varepsilon))^{\frac{q}{p-q}}} \le \begin{cases} -C_3\varepsilon^{\frac{p}{p-q} - \frac{pq}{p-q}} & \text{if } 1 < p \le 2, \\ -C_3\varepsilon^{\frac{p}{p-q} - \frac{2q}{p-q}} & \text{if } p \ge 2, \end{cases}
\end{aligned}$$

where $C_3 > 0$ is independent of ε. Since $p < 2q$, we get $M(\lambda^*) = -\infty$ by letting $\varepsilon \to 0$.

Assume now that $p \ge 2q$. We start by showing that $M(\lambda^*) > -\infty$. Suppose, by contradiction, that there exists a sequence $\{u_n\} \subset \mathcal{N}_{\lambda^*} \cap \mathcal{A}^+$ satisfying $J_{\lambda^*}(u_n) \to -\infty$ as $n \to +\infty$. For each n, we define the normalized function $w_n := u_n/\|\nabla u_n\|_p$. Since the fibered functional J_{λ^*} is 0-homogeneous, we have $J_{\lambda^*}(w_n) \to -\infty$. The boundedness of $\{w_n\}$ in $W_0^{1,p}$ implies its weak convergence in $W_0^{1,p}$ and strong convergence in $L^p(\Omega)$ to some w_0, up to a subsequence. Hence, recalling that $\lambda^* = \lambda_1(p)$, we get $0 \le E_{\lambda^*}(w_0) \le \liminf_{n\to+\infty} E_{\lambda^*}(w_n)$. On the other hand, the convergence $J_{\lambda^*}(w_n) \to -\infty$ and the boundedness of $\{w_n\}$ imply that $\liminf_{n\to+\infty} E_{\lambda^*}(w_n) = 0$, see (2.5). Consequently, $E_{\lambda^*}(w_0) = 0$, and hence $w_n \to w_0 = \varphi_p$ strongly in $W_0^{1,p}$. Applying Lemma 2.26, we deduce that $\{J_{\lambda^*}(w_n)\}$ has to be bounded from below, which gives a contradiction. In conclusion, in view of Remark 2.5, we have $M(\lambda^*) \in (-\infty, 0)$.

Finally, we prove that $M(\lambda^*)$ is attained provided $p > 2q$. Let $\{u_n\} \subset \mathcal{N}_{\lambda^*} \cap \mathcal{A}^+$ be a minimizing sequence for $M(\lambda^*)$. Let us show the boundedness of $\{u_n\}$. Suppose, contrary to our claim, that $\lim_{n\to+\infty} \|\nabla u_n\|_p = +\infty$ along a subsequence. Denoting, as above, $w_n := u_n/\|\nabla u_n\|_p$, we see that $w_n \to w_0 \in W_0^{1,p}$ weakly in $W_0^{1,p}$ and strongly in $L^p(\Omega)$, up to a subsequence. Since $\{u_n\} \subset \mathcal{N}_{\lambda^*} \cap \mathcal{A}^+$, each w_n satisfies

$$1 - \lambda^*\|w_n\|_p^p = E_{\lambda^*}(w_n) = \frac{1}{\|\nabla u_n\|_p^{p-q}}\int_\Omega a|w_n|^q\,dx.$$

Letting $n \to +\infty$ and recalling that $\lambda^* = \lambda_1(p)$, we get $w_0 \not\equiv 0$ a.e. in Ω and $E_{\lambda^*}(w_0) = 0$, whence $\{w_n\}$ converges to φ_p strongly in $W_0^{1,p}$. According to Lemma 2.26, we derive the following contradiction:

$$0 = \lim_{n\to+\infty} I_{\lambda^*}(t_{\lambda^*}(w_n)w_n) = \lim_{n\to+\infty} I_{\lambda^*}(u_n) = M(\lambda^*) < 0,$$

where we used the equality $t_{\lambda^*}(w_n)w_n = u_n$ by $u_n \in \mathcal{N}_{\lambda^*}$, $n \in \mathbb{N}$. That is, $\{u_n\}$ is bounded in $W_0^{1,p}$, and hence $\{u_n\}$ converges to some $u_0 \in W_0^{1,p}$ weakly $W_0^{1,p}$ and strongly in $L^p(\Omega)$, up to a subsequence. Since $E_{\lambda^*}(u_0) \geq 0$ in view of $\lambda^* = \lambda_1(p)$, the inequality $0 > M(\lambda^*) \geq I_{\lambda^*}(u_0)$ leads to $\int_\Omega a|u_0|^q\,dx > 0$, and hence $u_0 \notin \mathbb{R}\varphi_p$ because of the assumption $\int_\Omega a\varphi_p^q\,dx = 0$. Hence, the simplicity of $\lambda^* = \lambda_1(p)$ guarantees $E_{\lambda^*}(u_0) > 0$, and we get

$$M(\lambda^*) = \liminf_{n\to+\infty} I_{\lambda^*}(u_n) \geq I_{\lambda^*}(u_0) \geq I_{\lambda^*}(t_{\lambda^*}(u_0)u_0) \geq M(\lambda^*),$$

which means that u_0 is a minimizer of $M(\lambda^*)$. □

Proof of the assertion (iii). Since $\int_\Omega a\varphi_p^q\,dx > 0$, we have $\lambda^* = \lambda_1(p)$ by Proposition 2.1, and hence $E_{\lambda^*}(\varphi_p) = 0$. Fix any $\theta \in W_0^{1,p} \setminus \mathbb{R}\varphi_p$ and consider $w_\varepsilon := \varphi_p + \varepsilon\theta$ for $\varepsilon > 0$. The simplicity of $\lambda_1(p)$ ensures that $E_{\lambda^*}(w_\varepsilon) > 0$. Thus, we have $t_{\lambda^*}(w_\varepsilon)w_\varepsilon \in \mathcal{N}_{\lambda^*} \cap \mathcal{A}^+$ for any sufficiently small $\varepsilon > 0$, and

$$M(\lambda^*) \leq I_{\lambda^*}(t_{\lambda^*}(w_\varepsilon)w_\varepsilon) = J_{\lambda^*}(w_\varepsilon) = -\frac{p-q}{pq}\frac{\left(\int_\Omega a|w_\varepsilon|^q\,dx\right)^{\frac{p}{p-q}}}{E_{\lambda^*}(w_\varepsilon)^{\frac{q}{p-q}}} \to -\infty$$

as $\varepsilon \to 0$, since $\int_\Omega a|w_\varepsilon|^q\,dx \to \int_\Omega a\varphi_p^q\,dx > 0$ and $E_{\lambda^*}(w_\varepsilon) \to E_{\lambda^*}(\varphi_p) = 0$. □

3.2. Properties of the critical set of $\widetilde{M}(\lambda^*)$

Throughout this section, we always assume that either of the following two assumptions is satisfied:

(I) $\int_\Omega a\varphi_p^q\,dx < 0$.

(II) $\int_\Omega a\varphi_p^q\,dx = 0$, $p > 2q$, and $\partial\Omega$ is C^2-smooth and connected provided $N \geq 2$.

These assumptions coincide with the assumptions (I) and (II) of Theorem 1.16, respectively. In both cases, Theorem 1.11 asserts that $M(\lambda^*)$ is attained. Recall that $M(\lambda^*) = \widetilde{M}(\lambda^*)$, see Proposition 2.8.

In order to prove Theorem 1.16 (see Section 5), we need to establish several key properties of the set of minimizers of $\widetilde{M}(\lambda^*)$ (or, equivalently, the set of *nonnegative* minimizers of $M(\lambda^*)$):

$$K^* := \{u \in \widetilde{\mathcal{N}}_{\lambda^*} :\ \widetilde{I}_{\lambda^*}(u) = \widetilde{M}(\lambda^*)\}. \tag{3.8}$$

In particular, any $u \in K^*$ is a critical point of $\widetilde{I}_{\lambda^*}$.

Lemma 3.1. *K^* is a compact set. Moreover, for any $u \in K^*$ there holds*

$$\widetilde{E}_{\lambda^*}(u) = \int_\Omega a u_+^q\,dx = -\widetilde{M}(\lambda^*)\frac{pq}{p-q} > 0. \tag{3.9}$$

Proof. If $\lambda^* > \lambda_1(p)$ (which is the case of the assumption (I)), then the compactness of K^* follows from Lemma 2.22. Suppose that $\lambda^* = \lambda_1(p)$, that is, we are under the assumption (II). As in Lemma 2.22, in view of the (S_+)-property of the p-Laplacian, it is sufficient to establish the boundedness of an arbitrary sequence $\{u_n\} \subset K^*$. Let $\{u_n\}$ be such sequence. Suppose, by contradiction, that $\|\nabla u_n\|_p \to +\infty$ as $n \to +\infty$, up to a subsequence. Denoting $w_n := u_n/\|\nabla u_n\|_p$, we deduce from Lemma 2.24 that $w_n \to \varphi_p$ strongly in $W_0^{1,p}$. Moreover, we evidently have $t_{\lambda^*}(w_n)w_n = u_n \in \mathcal{N}_{\lambda^*} \cap \mathcal{A}^+$. Applying Lemma 2.26, we get the following contradiction:

$$0 > \widetilde{M}(\lambda^*) = \widetilde{I}_{\lambda^*}(u_n) = \widetilde{I}_{\lambda^*}(t_{\lambda^*}(w_n)w_n) \to 0 \quad \text{as } n \to +\infty.$$

Finally, the equality (3.9) directly follows from (2.6). $\square$

Let us now consider the closed δ-neighborhood of K^* with some $\delta > 0$:

$$K_\delta^* := \{u \in W_0^{1,p} : \ \mathrm{dist}(u, K^*) \le \delta\}, \tag{3.10}$$

where $\mathrm{dist}(u, K^*) := \inf\{\|\nabla(u - v)\|_p : v \in K^*\}$. We notice that the infimum is attained in view of the compactness of K^*. Hereinafter, by $B_\delta(u)$ we denote an open ball in $W_0^{1,p}$ of radius δ centered at u.

Lemma 3.2. *K_δ^* is weakly sequentially compact (and bounded). Moreover, if $\delta > 0$ is small enough, then there exists $C_1 > 0$ such that*

$$\widetilde{E}_{\lambda^*}(u) \ge C_1 \quad \text{and} \quad \int_\Omega a u_+^q \, dx \ge C_1 \quad \text{for any} \quad u \in K_\delta^*. \tag{3.11}$$

Proof. The boundedness simply follows from the compactness of K^*. Hence, any $\{u_n\} \subset K_\delta^*$ converges weakly in $W_0^{1,p}$ to some $u_0 \in W_0^{1,p}$, up to a subsequence. In view of the definition of K_δ^*, for any $n \in \mathbb{N}$ there exists $v_n \in K^*$ such that $\|\nabla(u_n - v_n)\|_p \le \delta$. Since K^* is compact, $\{v_n\}$ converges to some $v_0 \in K^*$ strongly in $W_0^{1,p}$, up to a subsequence. Thus, for any $\varepsilon > 0$ and any sufficiently large n, $v_n \in B_\varepsilon(v_0)$. Consequently, we have $u_n \in B_{\varepsilon+\delta}(v_0)$. Since $\overline{B_{\varepsilon+\delta}(v_0)}$ is weakly closed, we conclude that $u_0 \in \overline{B_{\varepsilon+\delta}(v_0)}$. Recalling that $\varepsilon > 0$ was arbitrary, we deduce that $u_0 \in \overline{B_\delta(v_0)}$, and hence $u_0 \in K_\delta^*$.

Let us prove the second part of the lemma. By the continuity and in view of Lemma 3.1, for every $v \in K^*$ there exists $\varepsilon(v) > 0$ such that

$$\widetilde{E}_{\lambda^*}(u) \ge -\frac{\widetilde{M}(\lambda^*)}{2}\frac{pq}{p-q} \quad \text{and} \quad \int_\Omega a u_+^q \, dx \ge -\frac{\widetilde{M}(\lambda^*)}{2}\frac{pq}{p-q} \tag{3.12}$$

for any $u \in \overline{B_{\varepsilon(v)}(v)}$. Since K^* is compact, there exist $v_1, \dots, v_k \in K^*$ such that $K^* \subset \cup_{j=1}^k B_{\varepsilon(v_j)/2}(v_j)$. Taking any $0 < \delta < \min\{\varepsilon(v_1), \dots, \varepsilon(v_k)\}/2$, we conclude that $K_\delta^* \subset \cup_{j=1}^k \overline{B_{\varepsilon(v_j)}(v_j)}$ since $\varepsilon(v_j)/2 + \delta \le \varepsilon(v_j)$ for each j. That is, (3.12) remains valid for any $u \in K_\delta^*$. $\square$

Finally, we consider the boundary of K_δ^*:

$$\partial K_\delta^* := \{u \in W_0^{1,p} : \ \mathrm{dist}(u, K^*) = \delta\}. \tag{3.13}$$

The main property of ∂K_δ^* is given in the following lemma which states that the elements of ∂K_δ^* have strictly higher energy than the elements of K^*.

Lemma 3.3. *Let $\delta > 0$ be small enough to satisfy (3.11). Then*

$$\widetilde{M}(\lambda^*) < \inf\{\widetilde{I}_{\lambda^*}(u) : \ u \in \partial K_\delta^* \}.$$

Proof. Let $\delta > 0$ be as required. We have

$$\widetilde{M}(\lambda^*) \le \widetilde{I}_{\lambda^*}(u) \quad \text{for any } u \in K^*_\delta. \tag{3.14}$$

Indeed, in view of (3.11), for any $u \in K^*_\delta$ there exists $t > 0$ such that $tu \in \widetilde{\mathcal{N}}_{\lambda^*}$ and $\widetilde{I}_{\lambda^*}(tu) < 0$ (cf. Section 2.2), and hence

$$\widetilde{M}(\lambda^*) \le \widetilde{I}_{\lambda^*}(tu) = \min_{s>0} \widetilde{I}_{\lambda^*}(su) \le \widetilde{I}_{\lambda^*}(u). \tag{3.15}$$

Suppose now, by contradiction to our main claim, that there exists a sequence $\{u_n\} \subset \partial K^*_\delta$ such that $\widetilde{I}_{\lambda^*}(u_n) \searrow \widetilde{M}(\lambda^*)$. Since K^*_δ is weakly sequentially compact by Lemma 3.2, $\{u_n\}$ converges to some $u_0 \in K^*_\delta$ weakly in $W_0^{1,p}$ and strongly in $L^p(\Omega)$, up to a subsequence. If $\|\nabla u_0\|_p < \liminf\limits_{n\to+\infty} \|\nabla u_n\|_p$, i.e., there is no strong convergence in $W_0^{1,p}$, then we get

$$\widetilde{I}_{\lambda^*}(u_0) < \liminf_{n\to+\infty} \widetilde{I}_{\lambda^*}(u_n) = \widetilde{M}(\lambda^*),$$

a contradiction to (3.14). Therefore, $\|\nabla u_0\|_p = \liminf\limits_{n\to+\infty} \|\nabla u_n\|_p$, which means that $u_n \to u_0$ strongly in $W_0^{1,p}$, and hence $u_0 \in \partial K^*_\delta$ $(\subset K^*_\delta)$. In view of (3.15), we obtain the chain

$$\widetilde{M}(\lambda^*) = \lim_{n\to+\infty} \widetilde{I}_{\lambda^*}(u_n) = \widetilde{I}_{\lambda^*}(u_0) \ge \min_{s>0} \widetilde{I}_{\lambda^*}(su_0) \ge \widetilde{M}(\lambda^*),$$

which implies that $u_0 \in \widetilde{\mathcal{N}}_{\lambda^*}$ and u_0 is a minimizer of $\widetilde{M}(\lambda^*)$. That is, $u_0 \in K^*$. Thus, we have $u_0 \in K^*$ and $u_0 \in \partial K^*_\delta$, simultaneously. But this is impossible since $K^* \cap \partial K^*_\delta = \emptyset$. $\square$

Remark 3.4. Any minimizer of $M(\lambda^*)$ is a local minimum point of I_λ (see Proposition 1.3), but we do not know a priori whether these minimizers are *strict* local minimum points. To put this less formally, Lemma 3.3 asserts that a neighborhood of the set of all nonnegative minimizers of $M(\lambda^*)$ has a strict local minimum type structure.

4. Qualitative properties of M and M^-

In this section, we establish several properties of the levels M and M^- and the corresponding minimizers. In particular, we prove Propositions 1.3, 1.6, and 1.9.

Proof of Proposition 1.3. Let u be a minimizer of $M(\lambda)$. Since $M(\lambda) < 0$ by Remark 2.5, we deduce from Lemma 2.6 that u is a solution of (P_λ). If u is not a ground state of (P_λ), then there exists a solution w such that $I_\lambda(w) < I_\lambda(u)$, which contradicts the definition of $M(\lambda)$ since $w \in \mathcal{N}_\lambda$. Let us show that u is a local minimum point of I_λ. Since $0 < E_\lambda(u) = \int_\Omega a|u|^q\,dx$, we can find a sufficiently small $r > 0$ such that $E_\lambda(v) > 0$ and $\int_\Omega a|v|^q\,dx > 0$ for any $v \in B_r(u)$, i.e., $\|\nabla(u-v)\|_p < r$. Thus, for each $v \in B_r(u)$ we have $t_\lambda(v)v \in \mathcal{N}_\lambda \cap \mathcal{A}^+$, and therefore

$$I_\lambda(u) = \inf\{I_\lambda(w) : w \in \mathcal{N}_\lambda\} \le I_\lambda(t_\lambda(v)v) = \min_{t>0} I_\lambda(tv) \le I_\lambda(v),$$

which means that u is a local minimizer of I_λ.

Let us prove that either $u > 0$ or $u < 0$ in Ω_a^+. Clearly, if u is a minimizer of $M(\lambda)$, then so is $v := |u|$. Hence v is a nonnegative local minimum point of I_λ, and we deduce from Lemma 2.11 that $v > 0$ in Ω_a^+. Consequently, u cannot change the sign inside any connected component of Ω_a^+.

Let us show that the minimizer u is a global minimum point of I_λ whenever $\lambda \leq \lambda_1(p)$. The case $\lambda < \lambda_1(p)$ is simple, see the discussion at the beginning of Section 1.1. Let $\lambda = \lambda_1(p)$ and suppose, by contradiction, that there exists w such that $I_\lambda(w) < I_\lambda(u)\,(< 0)$. Noting that $E_\lambda(w) \geq 0$ by the definition of $\lambda_1(p)$, we obtain $\int_\Omega a|w|^q\,dx > 0$. If $E_\lambda(w) > 0$, then $t_\lambda(w)w \in \mathcal{N}_\lambda \cap \mathcal{A}^+$ and $I_\lambda(t_\lambda(w)w) \leq I_\lambda(w)$, and hence we get a contradiction to the definition of $M(\lambda)$. If $E_\lambda(w) = 0$, then $w = t\varphi_p$ for some $t \neq 0$. That is, $\int_\Omega a\varphi_p^q\,dx > 0$. But in this case Theorem 1.11 (iii) yields $M(\lambda) = -\infty$, which is impossible since u is a minimizer of $M(\lambda)$ by the assumption. □

Proof of Proposition 1.6. (i) We start by showing that M is (strictly) decreasing on $(-\infty, \lambda^*]$. Recall that $M(\lambda)$ is attained for any $\lambda < \lambda^*$, see Theorem 1.5. Let u_λ be a corresponding minimizer, that is, $u_\lambda \in \mathcal{N}_\lambda \cap \mathcal{A}^+$ and $M(\lambda) = I_\lambda(u_\lambda)$.

Taking any $\lambda < \mu < \lambda^*$, we conclude from the definition (1.5) of λ^* that $E_\lambda(u_\lambda) > E_\mu(u_\lambda) > 0$. Therefore, $t_\mu(u_\lambda)u_\lambda \in \mathcal{N}_\mu \cap \mathcal{A}^+$, and we obtain the monotonicity:

$$
\begin{aligned}
M(\lambda) = I_\lambda(u_\lambda) = J_\lambda(u_\lambda) &= -\frac{p-q}{pq}\frac{\left(\int_\Omega a|u_\lambda|^q\,dx\right)^{\frac{p}{p-q}}}{(E_\lambda(u_\lambda))^{\frac{q}{p-q}}} \\
&> -\frac{p-q}{pq}\frac{\left(\int_\Omega a|u_\lambda|^q\,dx\right)^{\frac{p}{p-q}}}{(E_\mu(u_\lambda))^{\frac{q}{p-q}}} = J_\mu(u_\lambda) = I_\mu(t_\mu(u_\lambda)u_\lambda) \geq M(\mu).
\end{aligned}
$$

Let us show now that $M(\lambda) > M(\lambda^*)$ for any $\lambda < \lambda^*$. In view of the definition of λ^*, we have $E_{\lambda^*}(u_\lambda) \geq 0$. If $E_{\lambda^*}(u_\lambda) > 0$, then the same arguments as above yield the desired monotonicity. Assume that $E_{\lambda^*}(u_\lambda) = 0$. Recalling that $\int_\Omega a|u_\lambda|^q\,dx > 0$, we conclude that u_λ is a minimizer of λ^*, and hence Proposition 2.1 implies that $\lambda^* = \lambda_1(p)$ and $u_\lambda = t\varphi_p$ for some $t \neq 0$. Therefore, Theorem 1.11 (iii) gives $M(\lambda^*) = -\infty$, and the monotonicity of $M(\lambda)$ follows.

Finally, recalling that $M(\lambda) = -\infty$ for any $\lambda > \lambda^*$ (see Theorem 1.5), we deduce that M is nonincreasing on $\mathbb{R}$, which completes the proof.

(ii) The proof follows from Proposition 4.1 (i) below.

(iii) The proof follows from Proposition 4.1 (ii), (iii) below. □

Let us now prove the following general facts.

Proposition 4.1. *Let $\{\lambda_n\} \subset \mathbb{R}$ be a convergent sequence such that $\lambda_n < \lambda^*$, and set $\lambda := \lim\limits_{n\to+\infty} \lambda_n$. Let u_n be a ground state of (P_{λ_n}). Then the following assertions are satisfied:*

(i) *Let $\lambda < \lambda^*$. Then $\{u_n\}$ is bounded in $W_0^{1,p}$ and has a subsequence strongly convergent in $W_0^{1,p}$ to a ground state of (P_λ) as $n \to +\infty$.*

(ii) *Let $\lambda = \lambda^*$ and $M(\lambda^*) = -\infty$. Then $\lim\limits_{n\to+\infty} \|\nabla u_n\|_p = +\infty$, $\lim\limits_{n\to+\infty} I_{\lambda_n}(u_n) = -\infty$, $|u_n|/\|\nabla u_n\|_p \to \varphi_p$ strongly in $W_0^{1,p}$ as $n \to +\infty$ along a subsequence, and $\lambda^* = \lambda_1(p)$.*

(iii) *Let* $\lambda = \lambda^*$ *and* $M(\lambda^*) > -\infty$. *Then* $\lim_{n\to+\infty} I_{\lambda_n}(u_n) = M(\lambda^*)$.

Proof. Since $\lambda_n < \lambda^*$, the ground state u_n is a minimizer of $M(\lambda_n)$, and hence $|u_n|$ is also a ground state. Therefore, each $|u_n|$ is a solution of (P_{λ_n}) satisfying

$$E_{\lambda_n}(|u_n|) = \int_\Omega a|u_n|^q\,dx > 0. \tag{4.1}$$

Taking μ such that $\mu < \lambda = \lim_{n\to+\infty} \lambda_n$ and recalling that M is decreasing by Proposition 1.6 (i), we observe that

$$\limsup_{n\to+\infty} I_{\lambda_n}(u_n) = \limsup_{n\to+\infty} M(\lambda_n) \le M(\mu) < 0. \tag{4.2}$$

(i) Let $\lambda < \lambda^*$. We claim that $\{u_n\}$ is bounded in $W_0^{1,p}$. Suppose, by contradiction, that $\|\nabla u_n\|_p \to +\infty$. Clearly, we also have $\|\nabla|u_n|\|_p \to +\infty$. If $\lambda \neq \lambda_1(p)$, then the contradiction follows from Lemma 2.24 applied to $\{|u_n|\}$. In the case $\lambda = \lambda_1(p)$, Lemma 2.24 ensures that the sequence of $v_n := |u_n|/\|\nabla u_n\|_p$ converges to φ_p strongly in $W_0^{1,p}$, up to a subsequence. Since $\lim_{n\to+\infty} \int_\Omega a v_n^q\,dx \ge 0$ by (4.1), we have $\int_\Omega a\varphi_p^q\,dx \ge 0$, and hence $\lambda^* = \lambda_1(p)$ by Proposition 2.1. Thus, $\lambda < \lambda^* = \lambda_1(p)$, which contradicts our assumption $\lambda = \lambda_1(p)$. Consequently, $\{u_n\}$ is bounded in $W_0^{1,p}$, and Lemma 2.25 in combination with (4.2) guarantees that $\{u_n\}$ converges to a ground state of (P_λ) strongly in $W_0^{1,p}$, up to a subsequence.

(ii) If $\{u_n\}$ has a bounded subsequence, then, as above, Lemma 2.25 in combination with (4.2) justifies the existence of a ground state of (P_λ), and hence $M(\lambda^*) > -\infty$, which contradicts our assumption $M(\lambda^*) = -\infty$. Therefore, we have $\lim_{n\to+\infty} \|\nabla u_n\|_p = +\infty$, and Lemma 2.24 implies that the sequence of $|u_n|/\|\nabla u_n\|_p$ converges to φ_p strongly in $W_0^{1,p}$, up to a subsequence. Finally, let us show that $\lim_{n\to+\infty} I_{\lambda_n}(u_n) = -\infty$. Fix any $R > 0$. Since $M(\lambda^*) = -\infty$, we can choose $v \in \mathcal{N}_{\lambda^*} \cap \mathcal{A}^+$ such that $I_{\lambda^*}(v) \le -R$. Thus, for any sufficiently large n we have $t_{\lambda_n}(v)v \in \mathcal{N}_{\lambda_n} \cap \mathcal{A}^+$, and so

$$I_{\lambda_n}(u_n) = M(\lambda_n) \le I_{\lambda_n}(t_{\lambda_n}(v)v) \le I_{\lambda_n}(v).$$

This yields $\limsup_{n\to+\infty} I_{\lambda_n}(u_n) \le I_{\lambda^*}(v) \le -R$. Since $R > 0$ is arbitrary, we conclude that $\lim_{n\to+\infty} I_{\lambda_n}(u_n) = -\infty$.

(iii) We know from Proposition 1.6 (i) and Remark 2.5 that $0 > I_{\lambda_n}(u_n) \ge M(\lambda^*)$ for any $n \in \mathbb{N}$. Suppose, contrary to our claim, that there exists $C > 0$ such that $I_{\lambda_n}(u_n) \ge M(\lambda^*) + C$ for any n. Since $M(\lambda^*) > -\infty$, we can find $u \in \mathcal{N}_{\lambda^*} \cap \mathcal{A}^+$ satisfying

$$0 > I_{\lambda_n}(u_n) \ge M(\lambda^*) + C > I_{\lambda^*}(u) \ge M(\lambda^*). \tag{4.3}$$

Due to our assumptions $\lambda_n < \lambda^*$ and $u \in \mathcal{N}_{\lambda^*} \cap \mathcal{A}^+$, we have $E_{\lambda_n}(u) > E_{\lambda^*}(u) > 0$ for any n, and hence $t_{\lambda_n}(u)u \in \mathcal{N}_{\lambda_n} \cap \mathcal{A}^+$. This yields $I_{\lambda_n}(t_{\lambda_n}(u)u) \ge M(\lambda_n) = I_{\lambda_n}(u_n)$. On the other hand, by the continuity, we have $I_{\lambda_n}(t_{\lambda_n}(u)u) \to I_{\lambda^*}(u)$ as $n \to +\infty$. This contradicts (4.3) for all sufficiently large n. □

Proof of Proposition 1.9. For convenience, we prove our assertions in a nondirect order.

(v) Recall that $\lambda^* = \lambda_0^*$ and λ_0^* is attained, see Proposition 2.1. Let u_0 be a minimizer of λ_0^*. In particular, u_0 satisfies $E_{\lambda^*}(u_0) = 0 = \int_\Omega a|u_0|^q\,dx$. Fixing any $\lambda > \lambda^*$, we have

$E_\lambda(u_0) < 0$. Assume first that there exists $v \in W_0^{1,p}$ such that $\int_\Omega a|u_0|^{q-2}u_0 v\,dx < 0$, and set $u_\varepsilon := u_0 + \varepsilon v$ for $\varepsilon > 0$. Then for any sufficiently small $\varepsilon > 0$ we have

$$E_\lambda(u_\varepsilon) < 0 \quad \text{and} \quad \int_\Omega a|u_\varepsilon|^q\,dx = q\int_\Omega \int_0^\varepsilon a|u_s|^{q-2}u_s v\,ds\,dx < 0.$$

Consequently, $t_\lambda(u_\varepsilon)u_\varepsilon \in \mathcal{N}_\lambda \cap \mathcal{A}^-$ for such $\varepsilon > 0$, and we get

$$M^-(\lambda) = \inf_{u\in\mathcal{N}_\lambda\cap\mathcal{A}^-} I_\lambda(u) \le I_\lambda(t_\lambda(u_\varepsilon)u_\varepsilon) = J_\lambda(u_\varepsilon) = \frac{p-q}{pq}\frac{|\int_\Omega a|u_\varepsilon|^q\,dx|^{\frac{p}{p-q}}}{|E_\lambda(u_\varepsilon)|^{\frac{q}{p-q}}} \to 0 \tag{4.4}$$

as $\varepsilon \to 0$ because $E_\lambda(u_0) < 0$ and $\int_\Omega a|u_0|^q\,dx = 0$.

Assume now that $\int_\Omega a|u_0|^{q-2}u_0 v\,dx = 0$ for any $v \in W_0^{1,p}$. This implies that $a|u_0|^{q-2}u_0 \equiv 0$ a.e. in Ω, and hence $u_0 \equiv 0$ a.e. in $\Omega_a^\pm$. Take any $v \in C_0^\infty(\Omega)\setminus\{0\}$ with the support in Ω_a^- and set $u_\varepsilon := u_0 + \varepsilon v$. Recalling that $\int_\Omega a|u_0|^q\,dx = 0$, we obtain

$$\int_\Omega a|u_\varepsilon|^q\,dx = \int_\Omega a|u_0|^q\,dx + \varepsilon^q\int_\Omega a|v|^q\,dx = \varepsilon^q\int_\Omega a|v|^q\,dx < 0$$

for any $\varepsilon > 0$. Moreover, since $\lambda > \lambda^*$, we have $E_\lambda(u_\varepsilon) < 0$ for any sufficiently small $\varepsilon > 0$ by the continuity. Arguing as in (4.4) above, we obtain the desired result.

(i) Let $\lambda_1(p) < \lambda < \mu \le \lambda^*$ and let u_λ be a minimizer of $M^-(\lambda)$ which exists by Theorem 1.7. Noting that

$$E_\mu(u_\lambda) < E_\lambda(u_\lambda) = \int_\Omega a|u_\lambda|^q\,dx < 0,$$

we get $t_\mu(u_\lambda)u_\lambda \in \mathcal{N}_\mu \cap \mathcal{A}^-$, and hence

$$\begin{aligned} M^-(\lambda) = I_\lambda(u_\lambda) = J_\lambda(u_\lambda) &= \frac{p-q}{pq}\frac{|\int_\Omega a|u_\lambda|^q\,dx|^{\frac{p}{p-q}}}{|E_\lambda(u_\lambda)|^{\frac{q}{p-q}}} \\ &> \frac{p-q}{pq}\frac{|\int_\Omega a|u_\lambda|^q\,dx\,|^{\frac{p}{p-q}}}{|E_\mu(u_\lambda)|^{\frac{q}{p-q}}} = J_\mu(u_\lambda) = I_\mu(t_\mu(u_\lambda)u_\lambda) \ge M^-(\mu). \end{aligned}$$

That is, M^- is decreasing on $(\lambda_1(p), \lambda^*]$. The fact that M^- is nonincreasing on $(\lambda_1(p), +\infty)$ follows from the assertion (v).

(iii) Let $\{\lambda_n\}$ be any sequence satisfying $\lambda_n > \lambda_1(p)$ and $\lambda_n \to \lambda_1(p)$ as $n \to +\infty$. Choose a minimizer $u_n \ge 0$ of $M^-(\lambda_n)$, that is, $u_n \in \mathcal{N}_{\lambda_n} \cap \mathcal{A}^-$ and $M^-(\lambda_n) = I_{\lambda_n}(u_n)$. Consider the normalized function $v_n = u_n/\|\nabla u_n\|_p$. We have $E_{\lambda_n}(v_n) < 0$ and $\int_\Omega a v_n^q\,dx < 0$. Thanks to $\|\nabla v_n\|_p = 1$, there exists $v_0 \in W_0^{1,p}$ such that $v_n \to v_0$ weakly in $W_0^{1,p}$ and strongly in $L^p(\Omega)$, up to a subsequence. The inequality $E_{\lambda_n}(v_n) < 0$ implies $1 = \|\nabla v_n\|_p < \lambda_n\|v_n\|_p^p$, which yields $v_0 \not\equiv 0$ a.e. in Ω. Since $E_{\lambda_1(p)}(u) \ge 0$ for any $u \in W_0^{1,p}$ and $E_{\lambda_1(p)}(u) = 0$ if and only if $u = t\varphi_p$, $t \in \mathbb{R}$, we conclude that

$$0 \le E_{\lambda_1(p)}(v_0) \le \liminf_{n\to+\infty} E_{\lambda_n}(v_n) \le 0$$

and so $v_0 = t\varphi_p$ for some $t > 0$. Moreover, $v_n \to t\varphi_p$ strongly in $W_0^{1,p}$. Thus, we have

$$E_{\lambda_n}(v_n) \to t^p E_{\lambda_1(p)}(\varphi_p) = 0 \quad \text{and} \quad \int_\Omega a v_n^q\,dx \to t^q\int_\Omega a\varphi_p^q\,dx < 0,$$

where the last inequality is given by the assumption of the proposition. Consequently,

$$M^-(\lambda_n) = I_{\lambda_n}(u_n) = J_{\lambda_n}(u_n) = J_{\lambda_n}(v_n) = \frac{p-q}{pq} \frac{|\int_\Omega a v_n^q \, dx|^{\frac{p}{p-q}}}{|E_{\lambda_n}(v_n)|^{\frac{q}{p-q}}} \to +\infty,$$

which completes the proof since $\{\lambda_n\}$ is arbitrary.

(iv) Let us show that $M^-(\lambda) \to M^-(\lambda^*)$ as $\lambda \to \lambda^* - 0$. In view of the monotonicity stated in the assertion (i), we suppose, by contradiction, that there exists a sequence $\{\lambda_n\} \subset (\lambda_1(p), \lambda^*)$ such that $\lim\limits_{n\to+\infty} \lambda_n = \lambda^*$ and $M^-(\lambda^*) < \liminf\limits_{n\to+\infty} M^-(\lambda_n)$. Choose any $u \in \mathcal{N}_{\lambda^*} \cap \mathcal{A}^-$ such that

$$M^-(\lambda^*) \le I_{\lambda^*}(u) < \liminf_{n\to+\infty} M^-(\lambda_n). \tag{4.5}$$

By the continuity, we have $E_{\lambda_n}(u) \to E_{\lambda^*}(u) < 0$ as $n \to +\infty$. This yields $t_{\lambda_n}(u)u \in \mathcal{N}_{\lambda_n} \cap \mathcal{A}^-$ for any sufficiently large n, and $t_{\lambda_n}(u) \to 1$, and therefore

$$M^-(\lambda_n) \le I_{\lambda_n}(t_{\lambda_n}(u)u) \to I_{\lambda^*}(u) \quad \text{as } n \to +\infty,$$

which contradicts (4.5).

(ii) The continuity of $M^-(\lambda)$ follows from Proposition 4.3 below. □

Remark 4.2. The assertion (v) of Proposition 1.9 coincides with that of [41, Lemma 2.9 (2)]. This lemma is more general in nature, but its application to the problem (P_λ) requires additional assumptions on a (see [41, Section 3.1]). Arguing in much the same way as in the proof of the assertion (v) above, one can show that $M^-(\lambda) = 0$ for any $\lambda > \lambda_0^* \, (\ge \lambda^*)$, regardless the sign of $\int_\Omega a\varphi_p^q \, dx$.

Proposition 4.3. *Assume that $\int_\Omega a\varphi_p^q \, dx < 0$. Let $\{\lambda_n\} \subset \mathbb{R}$ be a convergent sequence such that $\lim\limits_{n\to+\infty} \lambda_n =: \lambda \in (\lambda_1(p), \lambda^*)$. Let u_n be a minimizer of $M^-(\lambda_n)$. Then $\{u_n\}$ is bounded in $W_0^{1,p}$ and it has a subsequence strongly convergent in $W_0^{1,p}$ to a minimizer $u_0 \in \mathcal{N}_\lambda \cap \mathcal{A}^-$ of $M^-(\lambda)$.*

Proof. Notice that $|u_n|$ shares with u_n the property of being a minimizer of $M^-(\lambda_n)$. In view of the assumption $\lambda > \lambda_1(p)$, we apply Lemma 2.24 to deduce the boundedness of $\{|u_n|\}$ and hence of $\{u_n\}$ in $W_0^{1,p}$. Thus, by Lemma 2.21, $\{u_n\}$ converges to a solution u_0 of (P_λ) strongly in $W_0^{1,p}$, up to a subsequence. We have $u_0 \not\equiv 0$ in Ω. Indeed, taking $\mu \in (\lambda, \lambda^*)$, Theorem 1.7 and Proposition 1.9 (i) imply that $M^-(\lambda_n) > M^-(\mu) > 0$ for any sufficiently large n. Therefore, we get

$$I_\lambda(u_0) = \lim_{n\to+\infty} I_{\lambda_n}(u_n) = \lim_{n\to+\infty} M^-(\lambda_n) \ge M^-(\mu) > 0,$$

whence $u_0 \in \mathcal{N}_\lambda \cap \mathcal{A}^-$ and, in particular, u_0 is nonzero. Finally, in order to prove that u_0 is a minimizer of $M^-(\lambda)$, let us show that $I_\lambda(u_0) \le I_\lambda(w)$ for all $w \in \mathcal{N}_\lambda \cap \mathcal{A}^-$. Fix any $w \in \mathcal{N}_\lambda \cap \mathcal{A}^-$. Then $t_{\lambda_n}(w)w \in \mathcal{N}_{\lambda_n} \cap \mathcal{A}^-$ for all sufficiently large n, and we obtain

$$I_{\lambda_n}(u_n) = M^-(\lambda_n) \le I_{\lambda_n}(t_{\lambda_n}(w)w). \tag{4.6}$$

Noting that $t_{\lambda_n}(w) \to 1$ by $w \in \mathcal{N}_\lambda$, and passing to the limit as $n \to +\infty$ in (4.6), we get our assertion. □

5. Existence after λ^*. Proof of Theorem 1.16

In this section, we prove Theorem 1.16. Throughout the section, *we always assume that either the assumption (I) or (II) of Theorem 1.16 is satisfied.*

5.1. First solution

We prove Theorem 1.16 (i) and the first part of Theorem 1.16 (ii) on the existence of a local minimum point.

5.1.1 Local minimizer

Our arguments will rely on the definitions and results from Section 3.2. Take a sufficiently small $\delta > 0$ as required in Lemmas 3.2 and 3.3. Let us show that there exists $\widehat{\lambda} > \lambda^*$ such that for any $\lambda \in [\lambda^*, \widehat{\lambda})$ we have

$$\inf\{\widetilde{I}_\lambda(u) : \; u \in K^*\} < \inf\{\widetilde{I}_\lambda(u) : \; u \in \partial K^*_\delta\}, \tag{5.1}$$

where the sets K^*, K^*_δ, ∂K^*_δ are defined by (3.8), (3.10), (3.13), respectively. We emphasize that these sets do not depend on λ. In the case $\lambda = \lambda^*$, (5.1) is stated in Lemma 3.3. The validity of (5.1) in a neighborhood of λ^* follows from the continuity arguments. More precisely, writing

$$\widetilde{I}_\lambda(u) = \widetilde{I}_{\lambda^*}(u) - \frac{\lambda - \lambda^*}{p} \int_\Omega u_+^p \, dx$$

and recalling that K^*_δ is bounded in $W_0^{1,p}$ by Lemma 3.2, we obtain the following uniform estimates with respect to $u \in K^*_\delta$ provided $\lambda \geq \lambda^*$:

$$\widetilde{I}_\lambda(u) \leq \widetilde{I}_{\lambda^*}(u) \leq \widetilde{I}_\lambda(u) + \frac{\lambda - \lambda^*}{p} \max\{\|v_+\|_p^p : \; v \in K^*_\delta\} = \widetilde{I}_\lambda(u) + (\lambda - \lambda^*)\, C.$$

Note that $C \in (0, +\infty)$ is independent of λ. Using Lemma 3.3, we get the following inequalities for any $\lambda \geq \lambda^*$:

$$\inf_{u \in K^*} \widetilde{I}_\lambda(u) \leq \inf_{u \in K^*} \widetilde{I}_{\lambda^*}(u) = \widetilde{M}(\lambda^*) < \inf_{u \in \partial K^*_\delta} \widetilde{I}_{\lambda^*}(u) \leq \inf_{u \in \partial K^*_\delta} \widetilde{I}_\lambda(u) + (\lambda - \lambda^*) C$$

Therefore, by the continuity, there exists $\widehat{\lambda} > \lambda^*$ such that (5.1) is satisfied for all $\lambda \in [\lambda^*, \widehat{\lambda})$.

Let us now consider a minimization problem

$$M_0(\lambda) := \inf\{\widetilde{I}_\lambda(u) : \; u \in K^*_\delta\}$$

for $\lambda \in (\lambda^*, \widehat{\lambda})$. Since K^*_δ is weakly sequentially compact (see Lemma 3.2), any minimizing sequence of $M_0(\lambda)$ has a weakly convergent subsequence and its weak limit u_λ belongs to K^*_δ. Noting that $\widetilde{I}_\lambda$ is weakly lower semicontinuous, we deduce that $M_0(\lambda)$ is attained by u_λ. In view of the inequality (5.1), u_λ stays in the interior of K^*_δ. Consequently, u_λ is a local minimum point of $\widetilde{I}_\lambda$ and hence a nonnegative solution of (P_λ). Clearly, u_λ is nonzero and $\widetilde{I}_\lambda(u_\lambda) < 0$ according to Lemma 2.14. Moreover, u_λ is positive in Ω_a^+ by Lemma 2.11.

We define a critical value

$$\Lambda^* := \sup \left\{ \bar{\lambda} > \lambda^* \;\middle|\; \begin{array}{l} \text{for any } \lambda \in (\lambda^*, \bar{\lambda}) \text{ there exist } u \in W_0^{1,p} \text{ and neighborhood} \\ K \text{ of } u \text{ such that } \widetilde{I}_\lambda(u) = \inf_K \widetilde{I}_\lambda < \inf_{\partial K} \widetilde{I}_\lambda \end{array} \right\}. \quad (5.2)$$

From the above arguments, we have $\Lambda^* > \lambda^*$. Notice that u in the definition of Λ^* is a local minimum point of $\widetilde{I}_\lambda$ which may not be a strict local minimum point. We conclude, as above, that u is a nonnegative solution of (P_λ) such that $\widetilde{I}_\lambda(u) < 0$ and $u > 0$ in Ω_a^+.

5.1.2 Least $\widetilde{I}_\lambda$-energy solution

Let us define a critical value

$$\Lambda := \sup\{\lambda : \ (P_\lambda) \text{ possesses a nonnegative solution } u \text{ such that } u > 0 \text{ in } \Omega_a^+\}. \quad (5.3)$$

We know from Section 5.1.1 that $\Lambda \geq \Lambda^* > \lambda^*$. Moreover, it will be proved in Section 5.3 that Λ is finite.

We start by showing that on the whole interval (λ^*, Λ), (P_λ) has a nonnegative solution which is positive in Ω_a^+. The interval (λ^*, Λ^*) is covered by Section 5.1.1. Thus, if $\Lambda = \Lambda^*$, then we are done. Assume that $\Lambda > \Lambda^*$ and take any $\lambda \in [\Lambda^*, \Lambda)$. By the definition of Λ, there exists $\bar{\lambda} \in (\lambda, \Lambda]$ for which $(P_{\bar{\lambda}})$ possesses a nonnegative solution $\bar{u}$ such that $\bar{u} > 0$ in Ω_a^+. Clearly, $\bar{u}$ is a *supersolution* of (P_λ), i.e.,

$$\int_\Omega |\nabla \bar{u}|^{p-2} \nabla \bar{u} \nabla \varphi \, dx \geq \lambda \int_\Omega \bar{u}^{p-1} \varphi \, dx + \int_\Omega a \bar{u}^{q-1} \varphi \, dx$$

for any nonnegative $\varphi \in W_0^{1,p}$. Moreover, recall that $\bar{u} \in L^\infty(\Omega) \cap C^{1,\beta}(\Omega)$ for some $\beta \in (0,1)$, see Remark 1.1. We take $\underline{u} = 0$ as a subsolution of (P_λ). It is not hard to verify that the set

$$\mathcal{S} := \{u \in W_0^{1,p} : \ 0 \leq u \leq \bar{u} \text{ a.e. in } \Omega\}$$

is closed in $W_0^{1,p}$ and convex, and hence $\mathcal{S}$ is weakly closed. Therefore, we deduce from [43, Theorem 1.2] that some $u_\lambda \in \mathcal{S}$ delivers a minimum value of I_λ (and hence of $\widetilde{I}_\lambda$) over $\mathcal{S}$. Then, arguing exactly as in the proof of [43, Theorem 2.4], we conclude that u_λ is a (nonnegative) solution of (P_λ).

Since u_λ is a minimizer of I_λ over $\mathcal{S}$, it is easily seen that that u_λ is nonzero. Indeed, consider any nonnegative $\varphi \in C_0^\infty(\Omega) \setminus \{0\}$ with the support in Ω_a^+. In view of the assumption $q < p$ and the inequality $\int_\Omega a\varphi^q \, dx > 0$, we deduce that $I_\lambda(t\varphi) < 0$ for any sufficiently small $t > 0$. Recalling that $\bar{u} > 0$ in Ω_a^+ and $\operatorname{supp} \varphi \subset \Omega_a^+$, we have $t\varphi \in \mathcal{S}$ provided $t > 0$ is small enough. Thus, $\min\{I_\lambda(w) : w \in \mathcal{S}\} < 0$, which yields $I_\lambda(u_\lambda) < 0$ and $u_\lambda \not\equiv 0$ in Ω.

Let us show that $u_\lambda > 0$ in Ω_a^+. We will argue similarly to the proof of Lemma 2.11. Suppose, contrary to our claim, that $u_\lambda(x_0) = 0$ for some $x_0 \in \Omega_a^+$. By the strong maximum principle, we have $u_\lambda \equiv 0$ in a connected component A of Ω_a^+ containing x_0. Consider any nonnegative $\varphi \in C_0^\infty(\Omega) \setminus \{0\}$ with the support in A. As above, since $q < p$ and $\int_\Omega a\varphi^q \, dx > 0$, we have $I_\lambda(t\varphi) < 0$ for any sufficiently small $t > 0$. Taking $t > 0$ smaller, if necessary, we also get $u_\lambda + t\varphi \in \mathcal{S}$. Therefore, we arrive at the following contradiction:

$$I_\lambda(u_\lambda) = \min\{I_\lambda(\omega) : \omega \in \mathcal{S}\} \leq I_\lambda(u_\lambda + t\varphi) = I_\lambda(u_\lambda) + I_\lambda(t\varphi) < I_\lambda(u_\lambda).$$

Since $\widetilde{I}_\lambda(u_\lambda) < 0$ and $\lambda > \lambda^* \geq \lambda_1(p)$, Proposition 2.23 guarantees the existence of the desired least $\widetilde{I}_\lambda$-energy solution w_λ for any $\lambda \in (\lambda^*, \Lambda)$, and $w_\lambda > 0$ at least in some connected component of Ω_a^+.

5.2. Mountain pass solution

In this section, we prove the existence of another solution v_λ of (P_λ) for any $\lambda \in (\lambda^*, \Lambda^*)$, as stated in Theorem 1.16 (ii), where Λ^* is defined by (5.2). First, we establish a slightly more general (but less precise) result.

Theorem 5.1. *Let $\lambda > \lambda^*$. Let $u \in W_0^{1,p}$ be a local minimum point of $\widetilde{I}_\lambda$. Then there exists another critical point v of $\widetilde{I}_\lambda$, $v \neq u$, $\widetilde{I}_\lambda(u) \leq \widetilde{I}_\lambda(v) < 0$, and $v > 0$ in some connected component of Ω_a^+.*

Proof. Let u be a local minimum point of $\widetilde{I}_\lambda$. Then $\widetilde{I}_\lambda(u) < 0$ by Lemma 2.14. Let us take any $\omega \in W_0^{1,p}$ satisfying $\widetilde{I}_\lambda(\omega) < \widetilde{I}_\lambda(u)$. Such ω exists since $\widetilde{M}(\lambda) = M(\lambda) = -\infty$ by Proposition 2.8, Theorem 1.5, and the assumption $\lambda > \lambda^*$. By standard arguments, we define the following mountain pass value:

$$c(\omega) := \inf_{\gamma \in \Gamma(\omega)} \max_{s \in [0,1]} \widetilde{I}_\lambda(\gamma(s)),$$

where

$$\Gamma(\omega) := \left\{ \gamma \in C([0,1], W_0^{1,p}) : \ \gamma(0) = u, \ \gamma(1) = \omega \right\}. \tag{5.4}$$

Since $\widetilde{I}_\lambda$ satisfies the Palais–Smale condition according to Lemma 2.22, [39, Theorem 1] implies that $c(\omega)$ is a critical level of $\widetilde{I}_\lambda$, and there exists a critical point $v = v(\omega)$ such that $c(\omega) = \widetilde{I}_\lambda(v)$ and $v \neq u$. On the one hand, if $c(\omega) = \widetilde{I}_\lambda(u)$, then [39, Theorem 1] ensures that v is also a local minimum point of $\widetilde{I}_\lambda$. Consequently, we have $c(\omega) = \widetilde{I}_\lambda(v) < 0$ and $v > 0$ in Ω_a^+ by Lemma 2.11, and we are done. On the other hand, if $c(\omega) > \widetilde{I}_\lambda(u)$, then v is a mountain pass solution of (P_λ). If we know that $c(\omega) < 0$, then $v \not\equiv 0$ in Ω_a^+, since otherwise Lemma 2.16 gives a contradiction to $\widetilde{I}_\lambda(v) = c(\omega) < 0$. That is, $v > 0$ at least in one connected component of Ω_a^+, see Remark 2.10. Thus, in order to conclude our proof, it suffices to show that $c(\omega) < 0$ for some function ω by constructing a "good" path in $\Gamma(\omega)$.

Recalling that $M(\lambda) = -\infty$ and passing to the absolute value, we have the existence of $\omega \in \widetilde{\mathcal{N}}_\lambda$ such that $\omega \geq 0$ a.e. in Ω and $\widetilde{I}_\lambda(\omega) < \widetilde{I}_\lambda(u)$. Consider the path

$$\xi(s) = ((1-s)u^q + s\omega^q)^{1/q} \quad \text{for } s \in [0,1].$$

Since $u, \omega \geq 0$ a.e. in Ω, we see that

$$\xi(s) \geq 0 \ \text{ a.e. in } \ \Omega, \quad \text{and so} \quad \widetilde{I}_\lambda(\xi(s)) = I_\lambda(\xi(s)) \ \text{ for any } s \in [0,1].$$

Moreover, by $\omega \in \widetilde{\mathcal{N}}_\lambda$ and $\widetilde{I}_\lambda(\omega) < 0$ we get $\int_\Omega a\omega^q \, dx > 0$, and hence

$$\int_\Omega a(\xi(s))^q \, dx = (1-s) \int_\Omega au^q \, dx + s \int_\Omega a\omega^q \, dx > 0 \ \text{ for any } s \in [0,1]. \tag{5.5}$$

If $\widetilde{E}_\lambda(\xi(s)) > 0$ for all $s \in [0,1]$, then we readily see that $\widetilde{I}_\lambda(t_\lambda(\xi(s))\xi(s)) < 0$ for all $s \in [0,1]$, and hence $t_\lambda(\xi(s\cdot))\xi(\cdot)$ is the desired path belonging to $\Gamma(\omega)$. Recalling that $\widetilde{E}_\lambda(\xi(0)) > 0$ and $\widetilde{E}_\lambda(\xi(1)) > 0$, we suppose now that there exists $s_0 \in (0,1)$ satisfying $\widetilde{E}_\lambda(\xi(s_0)) = 0$. Without loss of generality, we set

$$s_0 := \inf\{s \in (0,1) : \ \widetilde{E}_\lambda(\xi(s)) \leq 0\} = \inf\{s \in (0,1) : \ E_\lambda(\xi(s)) \leq 0\},$$

whence $\widetilde{E}_\lambda(\xi(s)) > 0$ for all $s \in (0, s_0)$. In view of (5.5), we deduce that $I_\lambda(t_\lambda(\xi(s))\xi(s)) < 0$ for all $s \in (0, s_0)$, and

$$I_\lambda(t_\lambda(\xi(s))\xi(s)) = J_\lambda(\xi(s)) = -\frac{p-q}{pq} \frac{\left(\int_\Omega a u^q \, dx\right)^{\frac{p}{p-q}}}{(E_\lambda(u))^{\frac{q}{p-q}}} \to -\infty \quad \text{as} \quad s \nearrow s_0.$$

Thus, we can find $s_1 \in (0, s_0)$ such that $\widetilde{I}_\lambda(t_\lambda(\xi(s_1))\xi(s_1)) = I_\lambda(t_\lambda(\xi(s_1))\xi(s_1)) < \widetilde{I}_\lambda(u)$. Taking $\widetilde{\omega} := t_\lambda(\xi(s_1))\xi(s_1)$ and considering the path $\eta(s) = t_\lambda(\xi(s_1 s))\xi(s_1 s)$ for $s \in [0, 1]$, we see that $\eta \in \Gamma(\widetilde{\omega})$ and $c(\widetilde{\omega}) \le \max_{s \in [0,1]} \widetilde{I}_\lambda(\eta(s)) < 0$, which completes the proof. $\square$

Now we are ready to prove the existence of the mountain pass solution of (P_λ) stated in Theorem 1.16 (ii). Let $\lambda \in (\lambda^*, \Lambda^*)$ and let u be a local minimum point of $\widetilde{I}_\lambda$ provided by the definition (5.2) of Λ^*. Recall that $\widetilde{I}_\lambda(u) < 0$ by Lemma 2.14. In view of the definition of Λ^*, there exists a neighborhood K of u with the following property:

$$\inf\{\widetilde{I}_\lambda(u') : \; u' \in \partial K\} > \inf\{\widetilde{I}_\lambda(u') : \; u' \in K\} = \widetilde{I}_\lambda(u). \tag{5.6}$$

Thus, any $\omega \in W_0^{1,p}$ such that $\widetilde{I}_\lambda(\omega) < \widetilde{I}_\lambda(u)$ must satisfy $\omega \notin \overline{K}$. Consequently, any path belonging to $\Gamma(\omega)$ (see (5.4)) intersects ∂K, and (5.6) yields $c(\omega) > \widetilde{I}_\lambda(u)$. Arguing now exactly as in the second part of the proof of Theorem 5.1, we obtain a mountain pass solution v_λ such that $0 > c(\omega)$(or $c(\widetilde{\omega})$) $= \widetilde{I}_\lambda(v_\lambda) > \widetilde{I}_\lambda(u)$, and $v_\lambda > 0$ in some connected component of Ω_a^+. This completes the proof of Theorem 1.16 (ii).

5.3. Boundedness of Λ. Proof of Theorem 1.16 (iii)

As a consequence of the definition (5.3) of Λ, for any $\lambda > \Lambda$ there is no nonnegative solution to (P_λ) which is positive in Ω_a^+. To make this statement nontrivial, we have to show that $\Lambda < +\infty$. In fact, we provide a slightly more general result, whose proof is rather standard anyway. Let us define

$$\Lambda_1 := \sup\{\lambda : \; (P_\lambda) \text{ possesses a solution } u \text{ such that } u > 0 \text{ in } \Omega_a^+\}.$$

That is, we do not require the solution u in the definition of Λ_1 to be nonnegative. Evidently, we have $\Lambda \le \Lambda_1$.

Proposition 5.2. *Assume that*

$$\lambda > \lambda_1(p; \Omega_a^+) := \inf\left\{ \frac{\int_{\Omega_a^+} |\nabla \varphi|^p \, dx}{\int_{\Omega_a^+} \varphi^p \, dx} : \; \varphi \in C_0^\infty(\Omega_a^+) \setminus \{0\}, \; \varphi \ge 0 \right\}.$$

Then (P_λ) does not possess a solution u satisfying $u > 0$ in Ω_a^+. In particular, $\Lambda \le \Lambda_1 \le \lambda_1(p; \Omega_a^+) < +\infty$.

Proof. Let $u \in W_0^{1,p}(\Omega)$ be a solution of (P_λ) with some $\lambda \in \mathbb{R}$ such that $u > 0$ in Ω_a^+. Recall that $u \in C_0^1(\Omega)$, see Remark 1.1. Let us take any nonnegative $\varphi \in C_0^\infty(\Omega) \setminus \{0\}$ satisfying $\operatorname{supp} \varphi \subset \Omega_a^+$. Then there exists $c > 0$ such that $u(x) \ge c$ for any $x \in \operatorname{supp} \varphi$. Therefore,

$\frac{\varphi}{u} \in L^\infty(\Omega)$, and so, by the regularity, $\frac{\varphi^p}{u^{p-1}} \in C_0^1(\Omega_a^+)$ and we can use it as a test function for (P_λ). Applying the standard Picone inequality [4, Theorem 1.1], we get

$$\begin{aligned} \int_{\Omega_a^+} |\nabla \varphi|^p \, dx &\geq \int_{\Omega_a^+} |\nabla u|^{p-2} \nabla u \nabla \left(\frac{\varphi^p}{u^{p-1}} \right) dx \\ &= \lambda \int_{\Omega_a^+} \varphi^p \, dx + \int_{\Omega_a^+} a(x) u^{q-p} \varphi^p \, dx \geq \lambda \int_{\Omega_a^+} \varphi^p \, dx. \end{aligned}$$

Consequently, λ must be bounded from above as follows:

$$\lambda \leq \frac{\int_{\Omega_a^+} |\nabla \varphi|^p \, dx}{\int_{\Omega_a^+} \varphi^p \, dx} < +\infty.$$

Minimizing over all such φ, we conclude that $\lambda \leq \lambda_1(p; \Omega_a^+)$. $\square$

Remark 5.3. The same argument as in the proof of Proposition 5.2 shows that (P_λ) *does not* possess a solution u satisfying $u > 0$ in a connected component A of Ω_a^+ provided $\lambda > \lambda_1(p; A)$. In particular, if $\{A_i\}$ is the set of all connected components of Ω_a^+ and $\max_i \lambda_1(p; A_i) < +\infty$, then (P_λ) does not possess a solution u satisfying $0 \not\equiv u \geq 0$ in Ω_a^+ whenever $\lambda > \max_i \lambda_1(p; A_i)$. Consequently, if (P_λ) has a nonzero solution u for such λ, then either $u \equiv 0$ in Ω_a^+ or there exists a connected component of Ω_a^+ where u changes sign.

6. Proof of Theorem 1.19

Let us note that $\lambda_1(p) = \lambda^*$ since we impose the assumption $\int_\Omega a \varphi_p^q \, dx = 0$, see Proposition 2.1. Recall that $\widetilde{I}_\lambda^\mu$ is the truncated functional defined by (1.9) with the weight function $a_\mu = a + \mu b$ replacing a.

The proof of the first part of Theorem 1.19 goes along the same lines as the proof of Theorem 1.16. Indeed, since $\widetilde{I}_\lambda^\mu$ is a continuous perturbation of $\widetilde{I}_{\lambda^*} (\equiv \widetilde{I}_{\lambda^*}^0)$, we have the analog of the key inequality (5.1) for $\widetilde{I}_\lambda^\mu$. Namely, there exist $\widehat{\mu} > 0$ and $\varepsilon > 0$ such that for any $\mu \in (-\widehat{\mu}, \widehat{\mu})$ and any $\lambda \in [\lambda_1(p) - \varepsilon, \lambda_1(p) + \varepsilon]$ we have

$$\inf\{\widetilde{I}_\lambda^\mu(u) : \ u \in K^*\} < \inf\{\widetilde{I}_\lambda^\mu(u) : \ u \in \partial K_\delta^*\}. \tag{6.1}$$

Here, K^*, K_δ^*, and ∂K_δ^* are the sets defined in Section 3.2, and these sets are independent of μ and λ. The case $\mu \leq 0$ implies $\int_\Omega a_\mu \varphi_p^q \, dx \leq 0$, which is covered by Theorem 1.16. That is why we are interested only in $\mu \in (0, \widehat{\mu})$, for which there holds $\int_\Omega a_\mu \varphi_p^q \, dx > 0$.

Fixing $\mu \in (0, \widehat{\mu})$ and considering a minimization problem

$$M_\mu(\lambda) := \inf\{\widetilde{I}_\lambda^\mu(u) : \ u \in K_\delta^*\}$$

for $\lambda \in [\lambda_1(p) - \varepsilon, \lambda_1(p) + \varepsilon]$, we deduce as in Section 5.1.1 that $M_\mu(\lambda)$ is attained by a critical point $u_\lambda \in K_\delta^*$ of $\widetilde{I}_\lambda^\mu$ which is a local minimum point, $\widetilde{I}_\lambda^\mu(u_\lambda) < 0$ by Lemma 2.14, and u_λ is positive in Ω_a^+ in view of Lemma 2.11. Arguing now exactly as in Sections 5.1.2, 5.2, and 5.3, we obtain the analogs of the assertions (i), (ii), (iii) of Theorem 1.16 for the problem (P_λ^μ) and the corresponding functional $\widetilde{I}_\lambda^\mu$.

Let us prove the second part of Theorem 1.19 on the existence of three solutions in a left neighborhood of $\lambda_1(p)$. Since $u_\lambda \in K_\delta^*$, K_δ^* is compact in $L^p(\Omega)$ and $L^q(\Omega)$ (see Lemma 3.2), and K_δ^* is independent of λ and μ, we obtain the following uniform lower bound on $\widetilde{I}_\lambda^\mu(u_\lambda)$:

$$\widetilde{I}_\lambda^\mu(u_\lambda) \geq -\frac{\lambda_1(p)}{p} \max\{\|v_+\|_p^p : v \in K_\delta^*\} - \frac{\|a\|_\infty + \widehat{\mu}\|b\|_\infty}{q} \max\{\|v_+\|_q^q : v \in K_\delta^*\}$$

for any $\lambda \in [\lambda_1(p) - \varepsilon, \lambda_1(p)]$. At the same time, in view of the inequality $\int_\Omega a_\mu \varphi_p^q \, dx > 0$, the global minimum point w_λ of $\widetilde{I}_\lambda^\mu$ given by Theorem 1.5 (i) for $\lambda < \lambda_1(p)$ satisfies $\widetilde{I}_\lambda^\mu(w_\lambda) \to -\infty$ as $\lambda \to \lambda_1(p) - 0$, see Proposition 1.6 (iii) in combination with Theorem 1.11 (iii). Thus, we conclude that there exists a sufficiently small $\epsilon > 0$ such that

$$\widetilde{I}_\lambda^\mu(w_\lambda) < \widetilde{I}_\lambda^\mu(u_\lambda) < 0 \quad \text{for any } \lambda \in (\lambda_1(p) - \epsilon, \lambda_1(p)). \tag{6.2}$$

In particular, w_λ is different from u_λ for such λ. Finally, using (6.2) and arguing as in Section 5.2 (see also [39, Corollary 1]), we establish the existence of the third critical point v_λ of $\widetilde{I}_\lambda^\mu$ for any $\lambda \in (\lambda_1(p) - \epsilon, \lambda_1(p))$. Thanks to the inequality (6.1), this critical point has the mountain pass type and satisfies $\widetilde{I}_\lambda^\mu(w_\lambda) < \widetilde{I}_\lambda^\mu(u_\lambda) < \widetilde{I}_\lambda^\mu(v_\lambda) < 0$. Therefore, we obtain three different nonnegative solutions of (P_λ). The proof is complete.

7. Nonexistence after λ^*. Proof of Theorem 1.23

Let $u \in W_0^{1,p}$ be a nonnegative solution of (P_λ) such that $u > 0$ in Ω_a^+. Fix any $\varepsilon > 0$. Then $\frac{\varphi_p}{u+\varepsilon} \in L^\infty(\Omega)$, and so we can choose $\frac{\varphi_p^q}{(u+\varepsilon)^{q-1}}$ as a test function for (P_λ). Applying the classical Picone inequality [4, Theorem 1.1] and the generalized Picone inequality [9, Theorem 1.8], we get

$$\begin{aligned} &\lambda \int_\Omega \left(\frac{u}{u+\varepsilon}\right)^{q-1} u^{p-q} \varphi_p^q \, dx + \int_\Omega a \left(\frac{u}{u+\varepsilon}\right)^{q-1} \varphi_p^q \, dx \\ &= \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \left(\frac{\varphi_p^q}{(u+\varepsilon)^{q-1}}\right) dx \leq \int_\Omega |\nabla \varphi_p|^{p-2} \nabla \varphi_p \nabla \left(\frac{\varphi_p^{q-p+1}}{(u+\varepsilon)^{q-p}}\right) dx. \end{aligned} \tag{7.1}$$

In order to simplify the most right integral in (7.1), let us investigate the regularity of the function $\varphi_p^{q-p+1}(u+\varepsilon)^{p-q}$. Since the assumption (1.10) is satisfied, we have $p < q+1$ by [9, Lemma 1.6]. Thus, recalling that $\varphi_p, u \in C^{1,\beta}(\overline{\Omega})$ (see Remark 1.1), we obtain $\varphi_p^{q-p+1}(u+\varepsilon)^{p-q} \in C(\overline{\Omega})$ and it is zero on $\partial\Omega$. On the other hand, we have

$$\left|\nabla \left(\frac{\varphi_p^{q-p+1}}{(u+\varepsilon)^{q-p}}\right)\right| \leq (q-p+1) \left(\frac{u+\varepsilon}{\varphi_p}\right)^{p-q} |\nabla \varphi_p| + (p-q) \left(\frac{\varphi_p}{u+\varepsilon}\right)^{q-p+1} |\nabla u|. \tag{7.2}$$

Clearly, the second term on the right-hand side of (7.2) is bounded in Ω. Noting that φ_p satisfies the Hopf maximum principle, we deduce the existence of a constant $c > 0$ such that $\varphi_p(x) \geq c \, \mathrm{dist}(x, \partial\Omega)$ for any $x \in \Omega$. This yields the following estimate for the integral of the first term on the right-hand side of (7.2):

$$(q-p+1) \int_\Omega \left(\frac{u+\varepsilon}{\varphi_p}\right)^{p-q} |\nabla \varphi_p| \, dx \leq C \int_\Omega \frac{1}{(\mathrm{dist}(x, \partial\Omega))^{p-q}} \, dx, \tag{7.3}$$

for some $C > 0$. Recalling that $p - q < 1$, we see that the integrals in (7.3) are bounded. Therefore, we conclude that $\varphi_p^{q-p+1}(u+\varepsilon)^{p-q} \in W_0^{1,1}(\Omega)$. Consequently, there exists a sequence $\{v_n\} \subset C_0^\infty(\Omega)$ such that $v_n \to \varphi_p^{q-p+1}(u+\varepsilon)^{p-q}$ strongly in $W^{1,1}(\Omega)$. By the triangle inequality, we get

$$\begin{aligned}
&\left| \int_\Omega |\nabla \varphi_p|^{p-2} \nabla \varphi_p \nabla \left(\frac{\varphi_p^{q-p+1}}{(u+\varepsilon)^{q-p}} \right) dx - \lambda_1(p) \int_\Omega \varphi_p^{p-1} \frac{\varphi_p^{q-p+1}}{(u+\varepsilon)^{q-p}} \, dx \right| \\
&\leq \left| \int_\Omega |\nabla \varphi_p|^{p-2} \nabla \varphi_p \nabla \left(\frac{\varphi_p^{q-p+1}}{(u+\varepsilon)^{q-p}} - v_n \right) dx \right| + \lambda_1(p) \left| \int_\Omega \varphi_p^{p-1} \left(\frac{\varphi_p^{q-p+1}}{(u+\varepsilon)^{q-p}} - v_n \right) dx \right| \\
&+ \left| \int_\Omega |\nabla \varphi_p|^{p-2} \nabla \varphi_p \nabla v_n \, dx - \lambda_1(p) \int_\Omega \varphi_p^{p-1} v_n \, dx \right|. \qquad (7.4)
\end{aligned}$$

Thanks to the $C^{1,\beta}(\overline{\Omega})$-regularity of φ_p and the convergence of $\{v_n\}$, the first and second integrals on the right-hand side of (7.4) tend to zero as $n \to \infty$. Moreover, since φ_p is the first eigenfunction of the p-Laplacian, the third integral on the right-hand side of (7.4) is zero for any n.

As a consequence, we deduce from (7.1) and (7.4) that

$$\lambda \int_\Omega \left(\frac{u}{u+\varepsilon} \right)^{q-1} u^{p-q} \varphi_p^q \, dx + \int_\Omega a \left(\frac{u}{u+\varepsilon} \right)^{q-1} \varphi_p^q \, dx \leq \lambda_1(p) \int_\Omega \varphi_p^{p-1} \frac{\varphi_p^{q-p+1}}{(u+\varepsilon)^{q-p}} \, dx.$$

Letting $\varepsilon \to 0$, we obtain

$$(\lambda_1(p) - \lambda) \int_\Omega u^{p-q} \varphi_p^q \, dx \geq \int_{\{x \in \Omega:\, u(x) > 0\}} a \varphi_p^q \, dx. \qquad (7.5)$$

Recalling that $u > 0$ in Ω_a^+ and $a \leq 0$ a.e. in $\Omega \setminus \Omega_a^+$ (the third assumption in (1.1)), we get

$$\int_{\{x \in \Omega:\, u(x) > 0\}} a \varphi_p^q \, dx = \int_\Omega a \varphi_p^q \, dx - \int_{\{x \in \Omega:\, u(x) = 0\}} a \varphi_p^q \, dx \geq \int_\Omega a \varphi_p^q \, dx. \qquad (7.6)$$

Combining (7.5) and (7.6), we arrive at

$$(\lambda_1(p) - \lambda) \int_\Omega u^{p-q} \varphi_p^q \, dx \geq \int_\Omega a \varphi_p^q \, dx.$$

This leads to either $\lambda < \lambda_1(p)$ or $\lambda \leq \lambda_1(p)$, provided $\int_\Omega a \varphi_p^q \, dx > 0$ or $\int_\Omega a \varphi_p^q \, dx = 0$, respectively, which proves the theorem.

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