ALTERNATIVE DEFINITION OF THE FIRST NONTRIVIAL FUČÍK CURVE

Consider the Fučik eigenvalue problem

$$\begin{cases} -\Delta_p u = \alpha (u^+)^{p-1} - \beta (u^-)^{p-1} & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $u = u^+ - u^-$, and $u^\pm := \max\{\pm u, 0\}$.

In [2] it is proved that the first nontrivial curve of the Fučik spectrum can be described as a set of points $(s + c(s), c(s))$, where $s \in \mathbb{R}$ and $c(s)$ defined by

$$c(s) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,1]} \left(\int_\Omega |\nabla u|^p \, dx - s \int_\Omega |u^+|^p \, dx \right).$$

Here

$$\Gamma := \{\gamma \in C([-1,1], S) : \ \gamma(-1) = -\varphi_1, \ \gamma(1) = \varphi_1\},$$

where $S := \{w \in W_0^{1,p} : \ \|w\|_{L^p} = 1\}$ and φ_1 is the first eigenfunction.

There is another characterization of the first nontrivial curve of the Fučík spectrum. Namely, consider

$$\alpha^*(\beta) := \inf \left\{ \frac{\int_\Omega |\nabla u^-|^p \, dx}{\int_\Omega |u^-|^p \, dx} : \ u \in W_0^{1,p}, \ u^\pm \not\equiv 0, \ \frac{\int_\Omega |\nabla u^+|^p \, dx}{\int_\Omega |u^+|^p \, dx} = \beta \right\}. \tag{2}$$

Note that the admissible set for this minimization problem is nonempty for all $\beta > \lambda_1(p)$. This definition is, in essence, the same as of Theorem 1.2 in [1] for the linear case $p = 2$ (see also [3]), and it was pointed out in this work that for $p > 1$ this definition is also ok. Let us prove this fact explicitly.

Proposition 0.1. *The set of points $(\alpha^*(\beta), \beta)$ is the first nontrivial curve of the Fučík spectrum.*

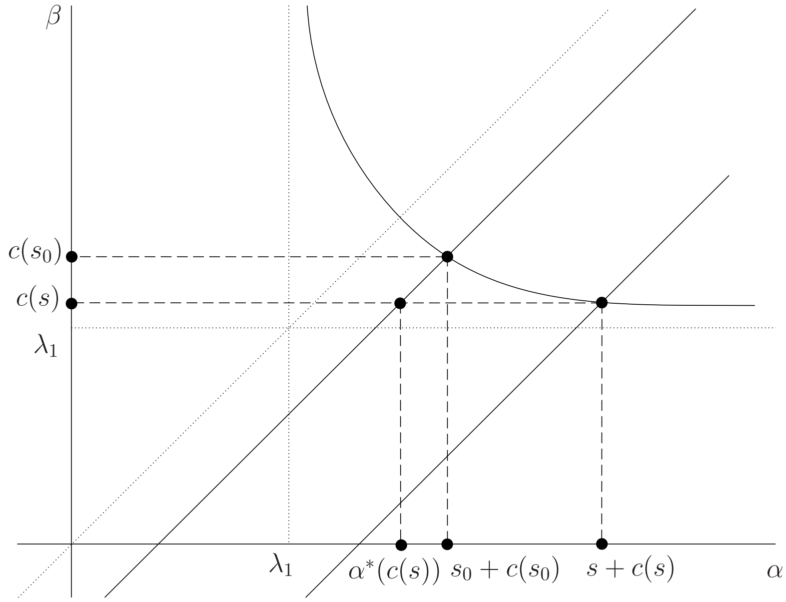
Proof. The main idea is to switch between the parametrizations: $c(s)$ parametrized by diagonals, while $\alpha^*(\beta)$ is parametrized by horizontal lines. Note that $c(s)$ is strictly decreasing [2, Propositions 4.1], i.e., $c(s) > c(s')$ whenever $s < s'$; moreover, $c(s) \to \lambda_1(p)$ as $s \to +\infty$, see [2, Proposition 4.4]. Thus, for each $\beta > \lambda_1(p)$ there exists unique $s \in \mathbb{R}$ such that $\beta = c(s)$. (see figure below). Notice that the $c(s)$ is constructed in [2] only for $s \geq 0$ and then the constructed part is reflected with respect to the bisector $\alpha = \beta$. However, it doesn't cause troubles.

Let us show now that $\alpha^*(c(s)) = s + c(s)$ for any $c(s) = \beta > \lambda_1(p)$. Note first that the eigenvalue which corresponds to $(\alpha, \beta) = (s + c(s), c(s))$ is always an admissible point for $\alpha^*(c(s))$, and hence $\alpha^*(c(s)) \leq s + c(s)$. Suppose, by contradiction, that $\alpha^*(c(s)) < s + c(s)$ for some s. Then, by definition of $\alpha^*(c(s))$, there have to exist a function $u \in W_0^{1,p}$ such that

$$\alpha^*(c(s)) \leq \frac{\int_\Omega |\nabla u^-|^p \, dx}{\int_\Omega |u^-|^p \, dx} < s + c(s) \quad \text{and} \quad \frac{\int_\Omega |\nabla u^+|^p \, dx}{\int_\Omega |u^+|^p \, dx} = \beta = c(s).$$

Due to the continuity and monotonicity of $c(s)$ [2, Proposition 4.1], there exists s_0 such that

$$\frac{\int_\Omega |\nabla u^-|^p \, dx}{\int_\Omega |u^-|^p \, dx} = s_0 + c(s) < s_0 + c(s_0) \quad \text{and} \quad \frac{\int_\Omega |\nabla u^+|^p \, dx}{\int_\Omega |u^+|^p \, dx} = \beta < c(s_0),$$



or, equivalently,

$$\int_\Omega |\nabla u^-|^p \, dx < (s_0 + c(s_0)) \int_\Omega |u^-|^p \, dx \quad \text{and} \quad \int_\Omega |\nabla u^+|^p \, dx < c(s_0) \int_\Omega |u^+|^p \, dx,$$

which is, in fact, the main contradictory assumption in the proof of [2, Theorem 3.1] (see also the proof of [2, Lemma 5.3, (5.10)]). Thus, proceeding exactly as in the proof of [2, Theorem 3.1], we obtain a contradiction to the definition of $c(s_0)$. □

References

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[2] Cuesta, M., De Figueiredo, D., & Gossez, J. P. (1999). The beginning of the Fucik spectrum for the p-Laplacian. Journal of Differential Equations, 159(1), 212-238.

[3] Molle, R., & Passaseo, D. (2015). Variational properties of the first curve of the Fučik spectrum for elliptic operators. Calculus of Variations and Partial Differential Equations, 54(4), 3735-3752.