HADAMARD SHAPE DERIVATIVE FORMULA FOR QUASILINEAR PROBLEMS. FORGOTTEN REFERENCES

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In a recently published (but submitted a long time ago) article [1] by S. Kolonitskii and myself, we studied the dependence of the least critical levels of the energy functional

$$E[u] = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \int_\Omega F(u) \, dx$$

upon domain perturbations driven by a family of diffeomorphisms

$$\Phi_t(x) = x + tR(x), \quad R \in C^1(\mathbb{R}^N, \mathbb{R}^N), \quad |t| < \delta. \tag{0.1}$$

Here F satisfies certain rather classical conditions.

Let us take an arbitrary minimizer v_0 of E over the Nehari manifold $\mathcal{N}(\Omega)$ and consider a function $v_t(y) := v_0(\Phi_t^{-1}(y))$, $y \in \Omega_t$. One of the main results of our paper is the following Hadamard-type formula:

$$\left.\frac{\partial E[\alpha(v_t)v_t]}{\partial t}\right|_{t=0} = -\frac{p-1}{p} \int_{\partial\Omega} \left|\frac{\partial v_0}{\partial n}\right|^p \langle R, n \rangle \, d\sigma, \tag{0.2}$$

where $\alpha(v_t) \in \mathbb{R}$ is a normalization coefficient such that $\alpha(v_t)v_t \in \mathcal{N}(\Phi_t(\Omega))$, and n is the outward unit normal vector to $\partial\Omega$.

In the particular case $F(u) = |u|^q$, $q \in [1, p^*)$, it can easily be checked that the finding of the least critical level of E can be restated as the finding of minimum for the problem

$$\mu_q(\Omega) = \min_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} J(u) := \min_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p \, dx}{\left(\int_\Omega |u|^q \, dx\right)^{\frac{p}{q}}}. \tag{0.3}$$

As a corollary of the proof of the Hadamard formula above, we obtain the following fact. If u_0 is a minimizer of $\mu_q(\Omega)$ normalized such that $\|u_0\|_{L^q(\Omega)} = 1$, and $u_t(y) := u_0(\Phi_t^{-1}(y))$, $y \in \Omega_t$, then

$$\left.\frac{\partial J(u_t)}{\partial t}\right|_{t=0} = -(p-1) \int_{\partial\Omega} \left|\frac{\partial u_0}{\partial n}\right|^p \langle R, n \rangle \, d\sigma. \tag{0.4}$$

Short historical remark. If $q = p$, then $\mu_p(\Omega)$ is the first eigenvalue of the p-Laplacian, and the formula (0.4) (in fact, the actual derivative of $\mu_p(\Omega)$) was established by García Melián & Sabina de Lis [4] in (2001) for $p > 1$, and by Lamberti [5] in (2003) for $p \geq 2$. In the case $p = 2$, this formula goes back to Hadamard (1908), see, e.g., [3].

However it appeared recently that the nonlinear case has been studied much earlier in a work of RopponGI [7] who established the Hadamard formula (0.4) for $p \geq 2$ and $q \geq p$ in (1994). In fact, the article of Ropponogi is build upon the article of Osawa [6] who considered the case $p = 2$, $q > 2$. Moreover, Osawa also treated the Hadamard formula under the Robin boundary condition and the Neumann boundary condition. The reader will

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be able to find easily a few more related works of this Japanese group by looking at citations of Osawa's article. Later, in (2017), the Hadamard formula (0.4) was independently stated in [2] by referring to the arguments from [4].

It is quite a pity that we didn't know these references before and didn't include them in our paper. Moreover, it seems that these works of the Japanese group are forgotten and have not being cited in related contemporary research. It makes sense to revive them.

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