HIGHER-ORDER VARIATIONS OF THE p-DIRICHLET ENERGY

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In this post, we would like to discuss some combinatorial aspects of the p-Laplacian. Namely, let $\int_\Omega |\nabla u|^p \, dx$ be the p-Dirichlet energy, where $u \in W^{1,p}(\Omega)$ and $p > 1$. Its first variation is given by

$$D^1 \left(\int_\Omega |\nabla u|^p \, dx \right) (\xi_1) = p \int_\Omega |\nabla u|^{p-2} (\nabla u, \nabla \xi_1) \, dx,$$

where $\xi_1 \in W^{1,p}(\Omega)$.

The second variation (if exists) is also easy to compute:

$$\begin{aligned} D^2 \left(\int_\Omega |\nabla u|^p \, dx \right) (\xi_1, \xi_2) &= p(p-2) \int_\Omega |\nabla u|^{p-4} (\nabla u, \nabla \xi_1)(\nabla u, \nabla \xi_2) \, dx \\ &\quad + p \int_\Omega |\nabla u|^{p-2} (\nabla \xi_1, \nabla \xi_2) \, dx, \end{aligned}$$

where $\xi_2 \in W^{1,p}(\Omega)$.

Let us make some effort to calculate the third variation:

$$\begin{aligned} D^3 \left(\int_\Omega |\nabla u|^p \, dx \right) (\xi_1, \xi_2, \xi_3) &= p(p-2)(p-4) \int_\Omega |\nabla u|^{p-6} (\nabla u, \nabla \xi_1)(\nabla u, \nabla \xi_2)(\nabla u, \nabla \xi_3) \, dx \\ &\quad + p(p-2) \int_\Omega |\nabla u|^{p-4} (\nabla u, \nabla \xi_1)(\nabla \xi_2, \nabla \xi_3) \, dx \\ &\quad + p(p-2) \int_\Omega |\nabla u|^{p-4} (\nabla u, \nabla \xi_2)(\nabla \xi_1, \nabla \xi_3) \, dx \\ &\quad + p(p-2) \int_\Omega |\nabla u|^{p-4} (\nabla u, \nabla \xi_3)(\nabla \xi_1, \nabla \xi_2) \, dx, \end{aligned}$$

where $\xi_3 \in W^{1,p}(\Omega)$.

We already start seeing some structure. So, let us now try to derive a general formula for the n-th variation of the energy functional. Our main result is the following one.

Proposition 0.1. *Let $u \in W^{1,p}(\Omega)$. If for a natural $n \geq 1$ there exists n-th variation of the p-Direchlet energy of u in direction $(\xi_1, \dots, \xi_n) \in (W^{1,p}(\Omega))^n$, then*

$$\begin{aligned} &D^n \left(\int_\Omega |\nabla u|^p \, dx \right) (\xi_1, \dots, \xi_n) \\ &= \int_\Omega \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} |\nabla u|^{p-2(n-i)} \prod_{j=0}^{n-i-1} (p-2j) \right. \end{aligned}$$

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$$\times \left[\sum_{\sigma \in B(n,n-2i)} \prod_{k=1}^{n-2i} (\nabla u, \nabla \xi_{\sigma(k)}) \left(\sum_{\omega \in P(n,\sigma)} \prod_{l=1}^{i} (\nabla \xi_{\omega(l,1)}, \nabla \xi_{\omega(l,2)}) \right) \right] \Bigg) dx,$$

where

(1) $B(n, n-2i)$ *is the set of all possible* $(n-2i)$*-combinations of* $\{1, 2, \dots, n\}$ *such that the ordering inside each* $\sigma \in B(n, n-2i)$ *is immaterial. Evidently, the cardinality of* $B(n, n-2i)$ *is* $\binom{n}{n-2i}$*. In particular, if* $i = 0$*, then* $card(B(n, n-2i)) = 1$*.*

(2) $P(n, \sigma)$ *is the set of all possible partitions of the set* $\{1, 2, \dots, n\} \setminus \sigma$ *into pairs such that the ordering of pairs and inside a pair is immaterial. Note that* $card(\sigma) = n-2i$*, and hence the number of pairs in each* $\omega \in P(\sigma)$ *is* i*. It is not hard to see that the cardinality of* $P(\sigma)$ *is* $\frac{(2i)!}{2^i i!}$*. We represent* ω *as a* $i \times 2$*-matrix* $(\omega(s,t))_{s=1..i,\ t=1,2}$*. For instance, if* $n = 6$ *and* $\sigma = \{1, 2\}$*, then*

$$P(\sigma) = \left\{ \begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 6 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 4 & 6 \end{pmatrix} \right\}.$$

Let us also calculate the fourth variation (just for fun) - either by straightforward computation, or by application of our general formula:

$$\begin{aligned}
&D^4 \left(\int_\Omega |\nabla u|^p \, dx \right) (\xi_1, \xi_2, \xi_3) \\
&= p(p-2)(p-4)(p-6) \int_\Omega |\nabla u|^{p-8} (\nabla u, \nabla \xi_1)(\nabla u, \nabla \xi_2)(\nabla u, \nabla \xi_3)(\nabla u, \nabla \xi_4) \, dx \\
&+ p(p-2)(p-4) \int_\Omega |\nabla u|^{p-6} (\nabla u, \nabla \xi_1)(\nabla u, \nabla \xi_2)(\nabla \xi_3, \nabla \xi_4) \, dx \\
&+ p(p-2)(p-4) \int_\Omega |\nabla u|^{p-6} (\nabla u, \nabla \xi_1)(\nabla u, \nabla \xi_3)(\nabla \xi_2, \nabla \xi_4) \, dx \\
&+ p(p-2)(p-4) \int_\Omega |\nabla u|^{p-6} (\nabla u, \nabla \xi_1)(\nabla u, \nabla \xi_4)(\nabla \xi_2, \nabla \xi_3) \, dx \\
&+ p(p-2)(p-4) \int_\Omega |\nabla u|^{p-6} (\nabla u, \nabla \xi_2)(\nabla u, \nabla \xi_3)(\nabla \xi_1, \nabla \xi_4) \, dx \\
&+ p(p-2)(p-4) \int_\Omega |\nabla u|^{p-6} (\nabla u, \nabla \xi_2)(\nabla u, \nabla \xi_4)(\nabla \xi_1, \nabla \xi_3) \, dx \\
&+ p(p-2)(p-4) \int_\Omega |\nabla u|^{p-6} (\nabla u, \nabla \xi_3)(\nabla u, \nabla \xi_4)(\nabla \xi_1, \nabla \xi_2) \, dx \\
&+ p(p-2) \int_\Omega |\nabla u|^{p-4} (\nabla \xi_1, \nabla \xi_2)(\nabla \xi_3, \nabla \xi_4) \, dx \\
&+ p(p-2) \int_\Omega |\nabla u|^{p-4} (\nabla \xi_1, \nabla \xi_3)(\nabla \xi_2, \nabla \xi_4) \, dx \\
&+ p(p-2) \int_\Omega |\nabla u|^{p-4} (\nabla \xi_1, \nabla \xi_4)(\nabla \xi_2, \nabla \xi_3) \, dx.
\end{aligned}$$

where $\xi_4 \in W^{1,p}(\Omega)$.

Visually, it could be easier to present this result as an n-th directional derivative of the p-th power of the norm of a vector. Namely, if $A, B_i \in \mathbb{R}^N$, then for any natural $n \geq 1$, we have

$$D^n\left(|A|^p\right)(B_1, \ldots, B_n)$$
$$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} |A|^{p-2(n-i)} \prod_{j=0}^{n-i-1} (p-2j) \left[\sum_{\sigma \in B(n,n-2i)} \prod_{k=1}^{n-2i} (A, B_{\sigma(k)}) \left(\sum_{\omega \in P(n,\sigma)} \prod_{l=1}^{i} (B_{\omega(l,1)}, B_{\omega(l,2)}) \right) \right].$$