PLEIJEL'S TYPE ESTIMATE FOR THE p-LAPLACIAN

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Consider the sequence $\{\lambda_n(\Omega)\}$ of eigenvalues of the Dirichelet p-Laplacian in a bounded domain $\Omega \subset \mathbb{R}^N$ obtained via the Lusternik–Schnirelmann min-max approach. Let φ_n be an eigenfunction associated to $\lambda_n(\Omega)$. We are interested in the estimates for the number of nodal domains of φ_n which we denote as $\mu(\varphi_n)$.

In the linear case $p = 2$, the well-known Courant nodal domain theorem says that $\mu(\varphi_n) \leq n$ for all $n \geq 1$. Its generalization to the nonlinear case $p \neq 2$ obtained in [1] asserts that

$$\mu(\varphi_n) \leq 2n - 2 \quad \text{for all } n \geq 2,$$

which implies

$$\limsup_{n\to\infty} \frac{\mu(\varphi_n)}{n} \leq 2.$$

On the other hand, in the linear case $p = 2$, there is a result of Pleijel [2] on the following asymptotic refinement of the Courant nodal domain theorem:

$$\limsup_{n\to\infty} \frac{\mu(\varphi_n)}{n} \leq \frac{4}{j_{0,1}^2} = 0.69166\ldots, \tag{1}$$

see, e.g., this post for a discussion.

The aim of the present post is to generalize the result of Pleijel to the p-Laplacian settings. Pleijel's approach is purely variational and consists of two main ingredients: *the Faber-Krahn inequality* and *the Weyl law*.

1. *The Faber-Krahn inequality* is easily available for the p-Laplacian, and it can be formulated as

$$|\Omega|^{\frac{p}{N}} \lambda_1(\Omega) \geq |B_1|^{\frac{p}{N}} \lambda_1(B_1),$$

where B_1 is a unit ball in $\mathbb{R}^N$; see, e.g., the discussion here. Therefore, noting that $\lambda_n(\Omega) = \lambda_1(\Omega_i)$ for any $i = 1..\mu(\varphi_n)$ where Ω_i is a nodal domain of φ_n, we get

$$|\Omega|\lambda_n(\Omega)^{\frac{N}{p}} \geq \mu(\varphi_n)|B_1|\lambda_1(B_1)^{\frac{N}{p}}.$$

Equivalently,

$$\mu(\varphi_n) \leq \frac{|\Omega|\lambda_n(\Omega)^{\frac{N}{p}}}{|B_1|\lambda_1(B_1)^{\frac{N}{p}}}. \tag{2}$$

2. *The Weyl law* is used to estimate $\lambda_n(\Omega)$ in (2) in terms of n. Unfortunately, this law is not available for the p-Laplacian in the required form; see the discussion in [3] and [4]. Instead, we will obtain the simplest explicit Weyl-type upper bound for $\lambda_n(\Omega)$, and this will be enough to get a Pleijel's type result. Let Q_h stands for the N-dimensional cube with the

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side length h. First, if $h \to 0$, then the number m of cubes Q_h disjointly inscribed in Ω is given by

$$m \approx \frac{|\Omega|}{h^N}.$$

Second, by the variational characterization of $\lambda_n(\Omega)$, we can estimate

$$\lambda_n(\Omega) \leq \lambda_1(Q_{h_n}),$$

where h_n is such that there are n disjoint cubes Q_{h_n} inscribed in Ω. We can assume that h_n is maximal.

Third, we know that

$$\lambda_1(Q_h) = \lambda_1(Q_1) h^{-p}.$$

Combining the previous three facts, we get

$$\lambda_n(\Omega) \leq \lambda_1(Q_{h_n}) = \lambda_1(Q_1) h_n^{-p} = \lambda_1(Q_1) \left(\frac{n}{|\Omega|}\right)^{\frac{p}{N}} \quad \text{as } n \to \infty. \tag{3}$$

Finally, mixing (2) and (3), we deduce that

$$\boxed{\limsup_{n\to\infty} \frac{\mu(\varphi_n)}{n} \leq \frac{1}{|B_1|} \left(\frac{\lambda_1(Q_1)}{\lambda_1(B_1)}\right)^{\frac{N}{p}}.} \tag{4}$$

Notice that this upper bound does not depend on Ω. Below, we will discuss a possible way how to improve this bound.

All we need now is to get a "good" upper bound for $\lambda_1(Q_1)$ and a "good" lower bound for $\lambda_1(B_1)$.

Let us start with an upper bound for $\lambda_1(Q_1)$. From Proposition 2.7 of [5] we know that

$$\lambda_1(Q_1) \leq \widetilde{\pi}_p^p N \quad \text{for} \quad p < 2$$

and

$$\lambda_1(Q_1) \leq \widetilde{\pi}_p^p N^{\frac{p}{2}} \quad \text{for} \quad p > 2,$$

where

$$\widetilde{\pi}_p = (p-1)^{\frac{1}{p}} \frac{2\pi}{p \sin(\pi/p)} \equiv 2(p-1)^{\frac{1}{p}} \int_0^1 \frac{ds}{(1-s^p)^{\frac{1}{p}}}.$$

As lower estimates for $\lambda_1(B_1)$, we use the estimate

$$\lambda_1(B_1) \geq N \left(\frac{p}{p-1}\right)^{p-1} \quad \text{for} \quad p < 2,$$

see [6] or [7]; and

$$\lambda_1(B_1) \geq Np \quad \text{for} \quad p > 2,$$

see [8] and, in general, this post for a discussion of lower bounds.

Thus, substituting all these things into (4), we get

$$\limsup_{n\to\infty} \frac{\mu(\varphi_n)}{n} \leq \frac{\Gamma\left(\frac{N}{2}+1\right) \pi^{\frac{N}{2}} 2^N (p-1)^N}{p^{\frac{(2p-1)N}{p}} \sin(\pi/p)^N} \quad \text{for} \quad p < 2 \tag{5}$$

and

$$\limsup_{n\to\infty} \frac{\mu(\varphi_n)}{n} \leq \frac{\Gamma\left(\frac{N}{2}+1\right)\pi^{\frac{N}{2}} 2^N N^{\frac{(p-2)N}{2p}} (p-1)^{\frac{N}{p}}}{p^{\frac{(p+1)N}{p}} \sin(\pi/p)^N} \quad \text{for} \quad p > 2. \tag{6}$$

The corresponding plot is depicted below by the increasing line. We see that these upper bounds does not give us a Pleijel constant smaller than 1 even in the dimension $N = 2$, which is quite sad. Note that if $p \to 1$, then the bound (5) approaches $\frac{4}{\pi} = 1.2732\ldots$, while if $p \to \infty$, then the bound (6) approaches $\frac{8}{\pi} = 2.5464\ldots$, see the blue line on figure below.

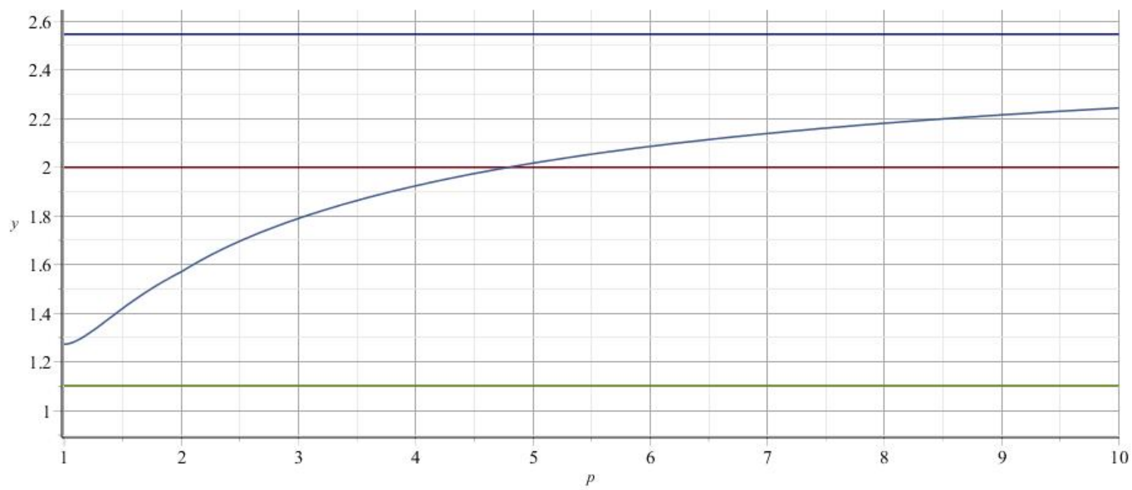

FIGURE 1. $N = 2$

Let us now discuss a possible improvement of (4) which concerns an improvement of the Weyl-type upper bound. For simplicity, let us fix $N = 2$. First, we can inscribe in Ω not a square tiling, but a hexagonal tiling. If H_r stands for a hexagon with the inradius r, and if $r \to 0$, then the number m of H_r's disjointly inscribed in Ω is given by

$$m \approx \frac{|\Omega|}{2\sqrt{3}r^2}.$$

Therefore, analogously to (3) we get

$$\lambda_n(\Omega) \leq \lambda_1(H_{r_n}) = \lambda_1(H_1) r_n^{-p} = \lambda_1(H_1)\left(\frac{2\sqrt{3}n}{|\Omega|}\right)^{\frac{p}{2}} \quad \text{as } n \to \infty,$$

and hence, from (2),

$$\boxed{\limsup_{n\to\infty} \frac{\mu(\varphi_n)}{n} \leq \frac{2\sqrt{3}}{|B_1|}\left(\frac{\lambda_1(H_1)}{\lambda_1(B_1)}\right)^{\frac{2}{p}}.} \tag{7}$$

Noting that $B_1 \subset H_1$, we get $\lambda_1(H_1) \leq \lambda_1(B_1)$, which yields

$$\limsup_{n\to\infty} \frac{\mu(\varphi_n)}{n} \leq \frac{2\sqrt{3}}{\pi} = 1.1026\ldots$$

Moreover, if $p \to \infty$, then, by the known result from [9], $\lambda_1(H_1)^{\frac{1}{p}} \to 1$ and $\lambda_1(B_1)^{\frac{1}{p}} \to 1$, i.e., this upper estimate of the upper estimate (7) is sharp for $p \to \infty$. See the green line on the figure above.

Thus, unfortunately, even if $n \to \infty$, we cannot show that $\mu(\varphi_n) \leq n$ for all $p > 1$ without getting a *substantial* improvement of the Weyl-type upper bound for $\lambda_n(\Omega)$. Such an improvement is clearly a prominent problem which needs to be studied much closer.

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