Pleijel's constant for the disk is $0.4613019\ldots$

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Consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain. Denote by $\{\lambda_i\}$ the sequence of the corresponding eigenvalues,

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \to \infty \quad \text{as } n \to \infty,$$

and let φ_n be an eigenfunction associated with λ_n. Let $\mu(\varphi_n)$ be a number of nodal domains of φ_n. Courant's theorem asserts that $\mu(\varphi_n) \leq n$ for any n. This result was refined by Å. Pleijel as follows.

Theorem 1 ([8, Section 5]). *Let $j_{0,1}$ be the first zero of the Bessel function J_0. Then*

$$Pl(\Omega) := \limsup_{n\to\infty} \frac{\mu(\varphi_n)}{n} \leq \frac{4}{j_{0,1}^2} = 0.69166\ldots$$

The upper bound $\frac{4}{j_{0,1}^2}$ is not sharp, as it was proved, e.g., by Bougain [2]. Moreover, it was conjectured by Polterovich [9] that

$$Pl(\Omega) \leq \frac{2}{\pi} = 0.63661\ldots$$

In fact, this conjectured upper bound is achieved for rectangles $\Omega = (0,a)\times(0,b)$ such that $\frac{a^2}{b^2}$ is irrational; see, e.g., [6]. However, it seems that, apart such rectangles, the Pleijel constant $Pl(\Omega)$ have not been found explicitly for any other domain Ω. At least, the question of finding such domains was explicitly posed by Bonnaillie-Noël et al in [1, Section 6.1].

The aim of the present note is to obtain the explicit expression for $Pl(B)$, where B is a unit disk (ball) in $\mathbb{R}^2$. Disk is the second most natural candidate for such tryings (after irrational rectangles), since we explicitly know all of its eigenvalues and eigenfunctions, and we know that its eigenfunctions have some good multiplicity properties. Our main result is the following.

Theorem 2.

$$Pl(B) = 8 \sup_{x>0} \left\{ x\,(\cos\theta(x))^2 \right\} = 0.4613019\ldots,$$

where $\theta = \theta(x)$ is the solution of the transcendental equation

$$\tan\theta - \theta = \pi x, \quad \theta \in \left(0, \frac{\pi}{2}\right).$$

Proof. Let $B := \{x \in \mathbb{R}^2 : \ |x| < 1\}$. By a separation of variables, it is not hard to see that any eigenfunction (up to rotation) can be expressed in the form

$$\varphi_{\nu,k}(r,\theta) = J_\nu(j_{\nu,k} r)\cos(\nu\theta), \quad \nu \in \mathbb{N}\cup\{0\},\ k \in \mathbb{N},$$

and $\lambda_{\nu,k} = j_{\nu,k}^2$ is the eigenvalue associated with $\varphi_{\nu,k}$. Here $j_{\nu,k}$ is the k-th zero of the ν-th Bessel function J_ν. Any eigenvalue $\lambda_{0,k}$ has multiplicity 1 (associated eigenfunction is radial), while any other eigenvalue has multiplicity 2 (associated eigenfunctions are $\varphi_{\nu,k}$ and its rotation). Clearly, $\mu(\varphi_{0,k}) = k$ and $\mu(\varphi_{\nu,k}) = 2\nu k$ for $\nu \in \mathbb{N}$.

Note that $\lambda_{\nu,k}$ is represented by two indexes ν, k, and it is not straightforwardly clear how to put $\lambda_{\nu,k}$ explicitly in the increasing order as λ_n. However, since we are interested in the behavior as $n \to \infty$, we use the Weyl law which, for $\Omega = B$, can be read as

$$n = \lambda_n \frac{|B|^2}{4\pi^2} + o(\lambda_n) = \frac{\lambda_n}{4} + o(\lambda_n).$$

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Hence, noting that for any λ_n there exists a unique pair (ν_n, k_n) such that $\lambda_n = \lambda_{\nu_n,k_n}$, we get

$$n = \frac{\lambda_{\nu_n,k_n}}{4} + o(\lambda_{\nu_n,k_n}) = \frac{j^2_{\nu_n,k_n}}{4} + o(j^2_{\nu_n,k_n}).$$

Recalling that $\mu(\varphi_{\nu,k}) = (2\nu + \sigma(\nu))k$, where $\sigma(0) = 1$ and $\sigma(\nu) = 0$ for $\nu \in \mathbb{N}$, we then deduce that

$$Pl(B) = \limsup_{n\to\infty} \frac{4(2\nu_n + \sigma(\nu_n))k_n}{j^2_{\nu_n,k_n}}.$$

Extracting a subsequence (still denoted by $\{n\}$) which delivers the value $Pl(B)$, omitting (for simplicity) subindex for (ν_n, k_n), and noting that $n \to \infty$ iff $\nu + k \to \infty$, we obtain

$$Pl(B) = \lim_{\nu+k\to\infty} \frac{4(2\nu + \sigma(\nu))k}{j^2_{\nu,k}}.$$

All we need now is to study the behavior of $j_{\nu,k}$ as $\nu + k \to \infty$. Let us immediately note that the sequence $\varphi_{0,k}$ cannot be a maximizing sequence for $Pl(B)$ since otherwise the inequality $j_{0,k} > k\pi - \frac{\pi}{4}$ (see [4, Eq. (1.2)]) yields

$$Pl(B) = \lim_{k\to\infty} \frac{4k}{j^2_{0,k}} \le \lim_{k\to\infty} \frac{4k}{\pi^2(k-1)^2} = 0,$$

but we will see later that $Pl(B) > 0$. Thus, we always assume that $\nu \in \mathbb{N}$, and hence $\mu(\varphi_{\nu,k}) = 2\nu k$ and

$$Pl(B) = \lim_{\nu+k\to\infty} \frac{8\nu k}{j^2_{\nu,k}}. \tag{1}$$

Note first that the following inequality is satisfies for all $\nu \ge 0$ and $k \in \mathbb{N}$:

$$j_{\nu,k} > \nu + \frac{\nu^{1/3}}{2^{1/3}} \left(\frac{3\pi}{8}(4k-1)\right)^{2/3}, \tag{2}$$

see the result of [4] or [10] in combination with the upper estimate for the zeros of the Airy function from [7, Theorem 2]. Therefore, estimating (2) from below by the first or the second summand, we deduce that

$$Pl(B) \le \lim_{\nu+k\to\infty} \min\left\{\frac{8k}{\nu}, C\left(\frac{\nu}{k}\right)^{1/3}\right\} \tag{3}$$

for some constant $C > 0$ which does not depend on ν and k.

Suppose at the moment that $Pl(B) > 0$. (We will achieve this fact later.) Under this assumption, we conclude from (3) that both ν and k tend to infinity, and there exist $A_1, A_2 > 0$ such that

$$A_1\nu < k < A_2\nu \quad \text{for all sufficiently large } \nu \in \mathbb{N}.$$

Moreover, recalling that (ν, k) is a maximizing subsequence for $Pl(B)$, we can always select a sub-subsequence (which is hence also a maximizing subsequence for $Pl(B)$) still denoted by (ν, k), such that

$$\lim_{\nu\to\infty} \frac{k}{\nu} = x_0 \in [A_1, A_2]. \tag{4}$$

That is, we have $k = \nu\, x_0 + o(k)$ for all large ν.

Let us now use the result of Elbert [3, Section 1.5] which states that

$$\lim_{\nu\to\infty} \frac{j_{\nu,\nu x}}{\nu} = \frac{1}{\cos\theta(x)}, \quad x > 0, \tag{5}$$

where $\theta = \theta(x)$ is the solution of the (transcendental) equation

$$\tan\theta - \theta = \pi x, \quad \theta \in \left(0, \frac{\pi}{2}\right). \tag{6}$$

Combining (1), (4), and (5), we see that $Pl(B) = 8x_0\,(\cos\theta(x_0))^2$, and x_0 have to satisfy

$$Pl(B) = 8x_0\,(\cos\theta(x_0))^2 = 8\sup_{x>0}\left\{x\,(\cos\theta(x))^2\right\} > 0. \tag{7}$$

Most likely, (6) and hence (7) cannot be solved in closed forms. However, one can convince himself that the left-hand side of (6) is strictly increasing in $\left(0, \frac{\pi}{2}\right)$, and hence the unique root of (6) and the value of $Pl(B)$ can be found with arbitrary precision via the standard numerical methods. In particular, using the build-in methods of *Mathematica*, we obtain

$$Pl(B) = 0.4613019\ldots \quad \text{and} \quad x_0 = \lim_{\nu\to\infty} \frac{k}{\nu} = 0.3710096\ldots$$

The corresponding *Mathematica* code can look like that:

```
F[x_?NumericQ] := y /. FindRoot[Tan[y] - y == Pi*x, {y, Pi/4}];
FindMaximum[8*x*(Cos[F[x]])^2, {x, 0.37}]
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□

Remark 3. The numerics suggests that the function $n \mapsto \frac{\mu(\varphi_n)}{n}$ decreases to $Pl(B)$ for large $n \in \mathbb{N}$. For instance,

$$\left.\frac{8\nu k}{j_{\nu,k}^2}\right|_{\nu=40000,\ k=0.371*40000=14840} = 0.4613096\ldots > 0.4613019\ldots = Pl(B).$$

Remark 4. If we consider the Neumann eigenvalues instead of the Dirichlet ones, then the result of our theorem remains valid. To show this fact, we argue in much the same way as in [5, Section 2.3]. Namely, Neumann eigenfunctions have the form

$$\psi_{\nu,k}(r,\theta) = J_\nu(j'_{\nu,k} r)\cos(\nu\theta), \quad \nu \in \mathbb{N} \cup \{0\},\ k \in \mathbb{N},$$

where $j'_{\nu,k}$ is the k-th zero of the derivative J'_ν of the Bessel function J_ν. Moreover, $\lambda_{\nu,k} = (j'_{\nu,k})^2$ is the associated eigenvalue. It is easy to see that, in fact, $\psi_{\nu,k}$ is a restriction to B of the Dirichlet eigenfunction $\varphi_{\nu,k}$ defined on a bigger ball B_R. Moreover,

$$R = \frac{j_{\nu,k}}{j'_{\nu,k}} \to 1 \quad \text{as } \nu \text{ or } k \to \infty,$$

and, clearly, $\mu(\phi_{\nu,k}) = 2\nu k$ for $\nu, k \in \mathbb{N}$. The result is then follows directly.

Remark 5. One can obtain similar explicit expressions of Pleijel's constant for some circular sectors, at least for those whose opening angle is π/m, $m \in \mathbb{N}$.

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