A remark on a lower bound for Neumann counting function à la Polya

Consider the Neumann eigenvalue problem

$$\begin{cases} -\Delta\, u = \mu u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded convex domain.

Recently, Filonov in [1] obtained the following lower bound on the eigenvalue counting function:

$$N_{\mathcal{N}}(\Omega, \mu) \geq \frac{|\Omega|\lambda}{2\sqrt{3}j_0^2},$$

where j_0^2 is the first positive zero of the Bessel function J_0.

Sketchily, the approach of [1] is the following: first we densely pack equal disks in $\mathbb{R}^2$, and then choose those whose centers lie in Ω. If a disk B is a subset of Ω, then we consider the first Dirichlet eigenfunction in B. If $B \setminus \Omega$ is nonempty, then we consider the restriction of the first Dirichlet eigenfunction in B to $\Omega \cap B$. Using these functions, we construct the test subspace and estimate $\mu_k(\Omega)$ from above by a factor coming from the number of disks and the first Dirichlet eigenvalue $\lambda_1(B)$, which leads to a required lower bound for $N_{\mathcal{N}}(\Omega, \mu)$.

The tricky point here is the consideration of the case when $B \setminus \Omega$ is nonempty. In this case, roughly speaking, one needs to justify that

$$\tau_1(\Omega \cap B) \leq \lambda_1(B),$$

where $\tau_1(\Omega \cap B)$ is the first eigenvalue in $\Omega \cap B$ under the zero Dirichlet boundary conditions on $\overline{\Omega} \cap \partial B$ and zero Neumann boundary conditions on the remaining part of $\partial(\Omega \cap B)$. This fact follows from Lemma 2.1 in [1] which states a certain integral property of Bessel functions. Here the convexity of Ω is employed. (Note that the fact remains true if Ω is merely star-shaped with respect to the center of B. However, it is hard to weaken the convexity in general, since the position of B with respect to Ω is not given constructively.)

It is tempting to anticipate that one could substitute disks by hexagons in the approach above, and hence improve the upper bound for $\mu_k(\Omega)$, thereby improving the lower bound for $N_{\mathcal{N}}(\Omega, \mu)$. To do it rigorously, one has to prove the inequality

$$\tau_1(\Omega \cap H) \leq \lambda_1(H),$$

where H is a hexagon such that $H \setminus \Omega \neq \emptyset$.

Unfortunately, it seems that the inequality cannot be true, in general. Let H be a hexagon with the side 1 centered at $(0, 0)$, and let Ω be a large triangle spanned on the points $(0, 0)$, $(-20, -2)$, $(20, -2)$, see Figure 1.

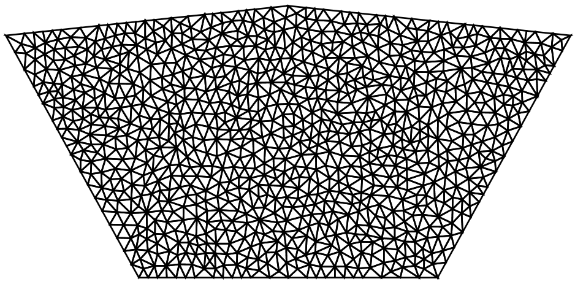

Figure 1. $\Omega \cap H$

Mathematica gives the following values for the corresponding eigenvalues:

$$\tau_1(\Omega \cap H) \approx 7.20569 \quad \text{and} \quad \lambda_1(\Omega) \approx 7.15548.$$

I admit that calculations might be completely wrong for $\tau_1(\Omega \cap H)$, but $\lambda_1(\Omega)$ is calculated more-less ok. Playing with parameters (vertices of the triangle Ω), I also observe continuous dependence of $\tau_1(\Omega \cap H)$ on them. The inequality holds for some parameters, and does not hold for others. This indirectly indicates that the calculation can be reliable.

Conclusion: it is not that easy to enhance the estimate from [1] using the same strategy.

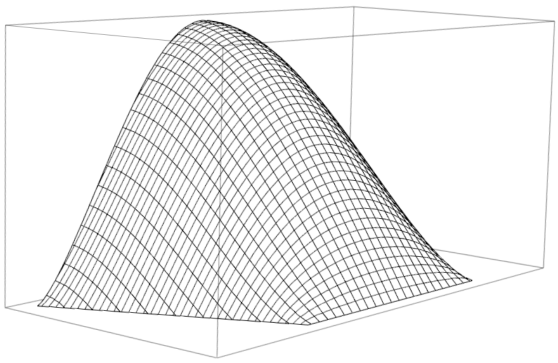

Figure 2. First mixed eigenfunction in $\Omega \cap H$

References

[1] Filonov, N. (2023). On the Pólya conjecture for the Neumann problem in planar convex domains. arXiv:2309.01432. 1, 2